Math 19
Midterm I Review

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Resources for Review

* Midterm I Review Guide
  http://www.courses.fas.harvard.edu:80/math19/exams/review1.pdf
* Exams and solutions from previous years http://www.courses.fas.harvard.edu/~math19/prevexams/

Exam Particulars

* Thursday, October 27 at 7-9 PM in Science Center A
* No calculators allowed
* All out-of-sequence exams must be approved
* No make-up exams

What to Expect

* Nine questions (some with several parts)
* The emphasis will be on material from Chapters 1-10
* Refer to the Midterm I Review Guide for details

Linearization

Consider the system of differential equations

$$\frac{dx}{dt} = x^2 - y^2 + 1,$$
$$\frac{dy}{dt} = -x.$$

(a) Draw and label the x and y null clines.

(b) Find the equilibrium points and label them on your drawing from part (a).

(c) Decide if the equilibrium points are stable. Justify your answer.

(d) On the drawing in part (a), label the regions where $dx/dt > 0$ and where $dx/dt < 0$. Do the same for dy/dt.

(e) What happens to

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

for small but positive t if $x(0) = 1$ and $y(0) = \sqrt{2}$?

Modeling

The rhinoceros is now extremely rare. Suppose that enough game preserve land is set aside so that there is sufficient room for many more rhinoceros territories than there are rhinoceros. Consequently, there will be no danger of overcrowding. However, if the population is too small, fertile adults have difficulty finding each other when it is time to mate. Write a differential equation that models the rhinoceros population based on these assumptions. (Note that there is more than one reasonable model that fits these assumptions).

Autonomous Differential Equations I

Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The population at $P = 0$ remains constant.
- A population close to 0 will decrease.
- A population of $P = 20$ will increase.
- A population of $P > 100$ will decrease.

Answer each of the following questions. (10 points)

(a) Sketch the simplest possible phase line that agrees with the experimental information above.

(b) Graph a rough sketch of the function $f(P)$ for the phase line of part (a).

Autonomous Differential Equations II

Consider the following differential equation for the function $y(t)$:

$$\frac{dy}{dt} = y^2 - 6y - 16.$$

(a) What are the equilibrium points?

(b) Which equilibrium points are stable?

(c) If $y(0) = 5$, what happens as t gets very large?

(d) If $y(0) = 10$, what happens as t gets very large?

Multivariable Calculus

Given $f(x, y) = y + \cos(x^2 y)$, compute the following partial derivatives:

(a) $\dfrac{\partial}{\partial x} f(x, y)$

(b) $\dfrac{\partial}{\partial y} f(x, y)$

(c) $\dfrac{\partial^2}{\partial x^2} f(x, y)$

Interacting Species

Consider the following two predator-prey systems of differential equations:

(i)

$$\begin{aligned}\frac{dx}{dt} &= 10x\left(1-\frac{x}{10}\right)-20xy,\\ \frac{dy}{dt} &= -5y+\frac{xy}{20}.\end{aligned}$$

(ii)

$$\begin{aligned}\frac{dx}{dt} &= 0.3x-\frac{xy}{100},\\ \frac{dy}{dt} &= 15y\left(1-\frac{y}{15}\right)+25xy.\end{aligned}$$

In one of these systems, the prey are very large animals and the predators are very small animals, such as elephants and mosquitos. Thus, it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey, such as whales and krill. Determine which system is which and provide a justification for your answer.

Linear Systems

Consider the system of differential equations

$$\frac{dx}{dt} = x - y,$$
$$\frac{dy}{dt} = 3y - 2x.$$

(a) Draw and label the x and y null clines.

(b) Label the equilibrium point and decide if it is stable. Justify your answer.

(c) On the drawing in part (a), label the regions where $dx/dt > 0$ and where $dx/dt < 0$. Do the same for dy/dt.

(d) For the initial condition $x(0) = 2$ and $y(0) = 1$, sketch the trajectory in the phase plane.