Math 19. Lecture 27
Traveling Wave Velocities

T. Judson

Fall 2005

1 The Model

We are looking for a solution $u(t, x)$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ru(1 - u), \tag{1}$$

where $r > 0$. In addition, we require that the following conditions be obeyed.

1. $0 \leq u \leq 1$,
2. $u(t, x) \to 1$ as $x \to -\infty$,
3. $u(t, x) \to 0$ as $x \to \infty$.

We examined traveling wave solutions of the form

$$u(t, x) = f(x - ct),$$

where $c > 0$. The constant c is the speed at which the wave described by f travels from left to right along the x-axis. If $u(t, x)$ satisfies the above equations, then $f(s)$ must obey

$$-c\frac{df}{ds} = \frac{d^2 f}{ds^2} + rf(1 - f), \tag{2}$$

where

1. $0 \leq f \leq 1$,

2. $f(s) \to 1$ as $s \to -\infty$,

3. $f(s) \to 0$ as $s \to \infty$.

We can turn this second-order differential equation such as (2) into a first-order system:

$$\frac{df}{ds} = p \tag{3}$$

$$\frac{dp}{ds} = -cp - rf(1-f). \tag{4}$$

We showed by phase plane analysis that this solution is solvable provided that $c^2 > 4r$.

2 The Problem

According to the preceding analysis, there are traveling wave solutions to (1) that cross country at arbitrarily high speeds. This seems contrary to our intuition.

3 The Solution

Consider the function $df/ds = p(s)$. Suppose that $p(s)$ has a maximum at some point, say at $s = s_0$. Then

$$\left.\frac{dp}{ds}\right|_{s=s_0} = 0.$$

Therefore, we can conclude from (4) that

$$cp(s_0) = -rf(s_0)(1 - f(s_0)). \tag{5}$$

Since $0 \leq f \leq 1$, we know that $df/ds = p \leq 0$. Now let s_1 be the point where $p(s)$ is the most negative. Then

$$\left.\frac{dp}{ds}\right|_{s=s_1} = 0,$$

and

$$-cp(s_1) = rf(s_1)(1 - f(s_1)). \tag{6}$$

Now, $0 \leq f(s_1) \leq 1$, which means that $f(s_1)(1 - f(s_1)) \leq 1/4$, since the parabola $y = x(1-x)$ has its maximum at $x = 1/2$, where $y = 1/4$. Therefore,

$$\max \left| \frac{df}{ds} \right| = -p(s_1) \leq \frac{r}{4c}. \tag{7}$$

This inequality needs some interpretation. It says that solutions to (2) with large c must have a small slope everywhere.

Thus, the velocity of the wave can be bounded everywhere if we know the maximum absolute value of the slope. That is, a high speed wave will have a very slow fall off in x, but a slow traveling wave will have a large fall off.

4 The Mouse/Virus Infection

In context of the mouse/virus problem, the maximum slope, as a function of x, of the fraction of the mice infected at time zero is the number

$$\max \left| \frac{\partial}{\partial x} u(0, x) \right|.$$

If we can determine this number from field data, then with a knowledge of the coefficient r, we can predict an upper bound to the speed at which the infection travels across country.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 22.