Math 19. Lecture 12
Matrix Notation

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1 Matrix Basics

Recall that we can rewrite the system

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

using matrix notation as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \frac{d\mathbf{x}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \text{ and } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

2 Matrix Operations

- Addition of matrices.
- Multiplying a matrix by a scalar.
- The product of a matrix and a column vector

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix}$$

- The product of two matrices

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} ex+fy \\ gx+hy \end{pmatrix} \\
&= \begin{pmatrix} (ae+bg)x+(af+bh)y \\ (ce+dg)x+(cf+dh)y \end{pmatrix} \\
&= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

- $AB = (A\mathbf{b}_1, A\mathbf{b}_2)$.
- In general, $AB \neq BA$.

3 Solving a 2×2 Linear System

Consider the system

$$\begin{aligned}
\frac{dx}{dt} &= x + 2y \\
\frac{dy}{dt} &= 4x + 3y
\end{aligned}$$

with initial conditions $x(0) = 1$ and $y(0) = 1$. For

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

we can guess that the solution is of the form

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix},$$

where $\lambda, c_1, c_2 \in \mathbb{R}$.

4 Checking the Guess

On the one hand, we have

$$\frac{d\mathbf{x}}{dt} = \lambda \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix} = \lambda \mathbf{x}.$$

On the other hand,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} = e^{\lambda t} A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To ensure a solution, we must choose λ, c_1, c_2 such that

$$A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

.

5 Finding Eigenvalues and Eigenvectors

Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The system $A\mathbf{x} = \lambda\mathbf{x}$ can be written as either

$$\begin{aligned} x_1 + 2x_2 &= \lambda x_1 \\ 4x_1 + 3x_2 &= \lambda x_2. \end{aligned}$$

We can reduce this system to

$$\begin{aligned} (1-\lambda)x_1 + 2x_2 &= 0 \\ (\lambda^2 - 4\lambda - 5)x_2 &= 0. \end{aligned}$$

Therefore, to obtain a nonzero solution either $\lambda = 5$ or $\lambda = -1$.

- If $\lambda = 5$, the first equation in the system becomes $-2x_1 + x_2 = 0$, and we can let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- If $\lambda = -1$, then

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

  For

$$A\mathbf{x} = \lambda\mathbf{x},$$

The number λ is called an *eigenvalue* of A, and $\mathbf{x} \neq \mathbf{0}$ is an *eigenvector* corresponding to λ.

6 The Principle of Superposition

The solutions to the system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

must be of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \beta e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The *Principle of Superposition* tells us that any linear combination of solutions to a linear equation is also a solution. Therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} x(t) &= \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} \\ y(t) &= \frac{4}{3}e^{5t} - \frac{1}{3}e^{-t}. \end{aligned}$$

Homework

- Chapter 11. Exercises 1, 2, 3, 4; p. 177.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 11.