

Math 19. Mathematical Modeling. Solutions to Exam II

T. Judson

Fall 2002

1. For each of the following cases, indicate whether modeling with an advection equation, a diffusion equation, or Laplace's equation is more appropriate. (10 points)

- (a) The concentration of a drug in the bloodstream after it is injected into the arm.

Solution. Advection.

- (b) The distribution of a certain protein in a cell after the protein has reached a steady-state.

Solution. Laplace.

- (c) The spread of airborne radioactivity after Chernobyl disaster in 1986.

Solution. Advection.

- (d) The spread of the oil spill from the Exxon Valdez in Prince William Sound in 1989.

Solution. Advection.

- (e) The spread of killer bees when they first escaped from captivity in Brazil.

Solution. Diffusion.

2. Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \quad (1)$$

(15 points)

- (a) Verify that $u(t, x) = e^{-t} \sin x$ is a solution to (1).

Solution. Differentiating u with respect to t , we get

$$\frac{\partial u}{\partial t} = -e^{-t} \sin x.$$

On the other hand, differentiating u with respect to x , we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} e^{-t} \sin x = e^{-t} \frac{\partial^2}{\partial x^2} \sin x = -e^{-t} \sin x.$$

Therefore, $u(t, x) = e^{-t} \sin x$ is a solution to (1).

- (b) Verify that $u(t, x) = e^{-4t} \sin 2x$ is a solution to (1).

Solution. Differentiating u with respect to t , we get

$$\frac{\partial u}{\partial t} = -4e^{-4t} \sin 2x.$$

On the other hand, differentiating u with respect to x , we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} e^{-4t} \sin 2x = e^{-4t} \frac{\partial^2}{\partial x^2} \sin 2x = -4e^{-4t} \sin 2x.$$

Therefore, $u(t, x) = e^{-4t} \sin 2x$ is a solution to (1).

- (c) Use these functions and the Principle of Superposition to construct a solution to (1) that satisfies the boundary and initial conditions

$$\begin{aligned} u(0, x) &= 20 \sin x + 10 \sin 2x, \\ u(t, 0) &= 0. \end{aligned}$$

Solution. Using the Principle of Superposition

$$u(t, x) = \alpha e^{-t} \sin x + \beta e^{-4t} \sin 2x$$

is a solution to (1) for all real numbers α and β . This solution certainly satisfies the initial condition $u(t, 0) = 0$. If we choose $\alpha = 20$ and $\beta = 10$, then $u(0, x) = 20 \sin x + 10 \sin 2x$ is satisfied.

3. Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u. \tag{2}$$

(20 points)

- (a) Use the separation of variables technique to find all nonzero solutions to (2) subject to the conditions for all t and for $0 \leq x \leq L$ such that $u(t, 0) = u(t, L) = 0$. [*Hint:* Look for solutions of the form $u(t, x) = A(t)B(x)$.]

Solution. If $u(t, x) = A(t)B(x)$, then

$$B \frac{dA}{dt} = A \frac{d^2 B}{dx^2} + 5AB$$

or

$$\frac{1}{A} \frac{dA}{dt} = \frac{1}{B} \frac{d^2 B}{dx^2} + 5 = \lambda$$

for some constant λ . Solving

$$\frac{dA}{dt} = \lambda A,$$

we have $A(t) = A(0)e^{\lambda t}$. For

$$\frac{1}{B} \frac{d^2 B}{dx^2} + 5 = \lambda \text{ or } \frac{d^2 B}{dx^2} = cB,$$

where $c = \lambda - 5$, the boundary conditions cannot be satisfied unless $c < 0$.

- If $c = 0$, then $B(x) = \alpha + \beta x$. The boundary conditions tell us that $\alpha = \beta = 0$. Therefore, we have no nonzero solutions if $c = \lambda - 5 = 0$.
- If $c > 0$, then $B(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$. The boundary conditions tell us that

$$\begin{aligned} B(0) &= \alpha + \beta = 0 \\ B(L) &= \alpha e^{\sqrt{c}L} + \beta e^{-\sqrt{c}L} = 0. \end{aligned}$$

Therefore, $\beta = -\alpha$ and

$$\alpha(e^{\sqrt{c}L} - e^{-\sqrt{c}L}) = 0.$$

Since $e^{\sqrt{c}L} - e^{-\sqrt{c}L} > 0$ for $L > 0$, It must be the case that $\alpha = 0$. Thus, we have no nonzero solutions if $c = \lambda - 5 > 0$.

- If $c < 0$, then

$$B(x) = \alpha \cos(\sqrt{-c}x) + \beta \sin(\sqrt{-c}x).$$

The boundary condition $B(0) = 0$ tells us that $\alpha = 0$. The second boundary condition, $B(L) = 0$, says that

$$0 = B(L) = \beta \sin(\sqrt{-c}L).$$

Therefore, we have a nonzero solution if $\sqrt{-c} = n\pi/L$ or $c = \lambda - 5 = -n^2\pi^2/L^2$ for any integer n . So the nonzero solutions of (2) are

$$u(t, x) = \alpha e^{\lambda t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\lambda = 5 - \frac{n^2\pi^2}{L^2}.$$

- (b) Find the minimum positive number L with the following property:
There is a solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u$$

for all t and for $0 \leq x \leq L$ such that $u(t, 0) = u(t, L) = 0$ and such that grows in size as $t \rightarrow \infty$.

Solution.

$$u(t, x) = \alpha e^{\lambda t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\lambda = 5 - \frac{n^2\pi^2}{L^2}.$$

to grow without bound, it is necessary that we choose $\lambda > 0$ or $L^2 > n^2\pi^2/5$. The smallest such number occurs when $n = 1$ or $L > \pi/\sqrt{5}$.

4. Consider the equilibrium solution $u_e = -1$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 - u$$

satisfying the boundary conditions

$$\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, L) = 0.$$

Determine the stability of this solution. (10 points)

Solution. Since $f(u) = u^2 - u$, we know that $f'(-1) = -3$. Therefore, we must determine if there is a $\lambda \geq 0$ such that

$$\lambda g = \frac{d^2 g}{dx^2} - 3g \tag{3}$$

has a solution for g that is not identically zero. If we let $c = \lambda + 3$, then this equation becomes

$$\frac{d^2 g}{dx^2} = cg$$

subject to the boundary conditions

$$\left. \frac{dg}{dx} \right|_{x=0} = \left. \frac{dg}{dx} \right|_{x=L} = 0.$$

We have the usual three cases $c < 0$, $c = 0$, and $c > 0$

- If $c = \lambda + 3 < 0$, then $\lambda < -3$, and we can find no pair (λ, g) .
- If $c = \lambda + 3 = 0$, then $\lambda = -3$, and we can find no pair (λ, g) .

- If $c > 0$, then

$$\begin{aligned} g(x) &= \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x} \\ \frac{dg}{dx} &= \alpha\sqrt{c}e^{\sqrt{c}x} - \beta\sqrt{c}e^{-\sqrt{c}x}. \end{aligned}$$

The boundary condition at $x = 0$ tells us that $\alpha = \beta$. Therefore, at $x = l$, we know that

$$0 = \left. \frac{dg}{dx} \right|_{x=L} = \alpha\sqrt{c}e^{\sqrt{c}L} - \alpha\sqrt{c}e^{-\sqrt{c}L} = \alpha\sqrt{c}(e^{\sqrt{c}L} - e^{-\sqrt{c}L}).$$

Since $e^{\sqrt{c}L} > e^{-\sqrt{c}L}$ for $L > 0$, it must be the case that $\alpha = 0$. Therefore, the solution to (3) is $g \equiv 0$.

Since we can find no pair (λ, g) , the solution $u_e = -1$ is stable.

5. Let

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u),$$

where

$$\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, L) = 0.$$

If u has no time derivative, then u is a function only of x , i.e. $u(t, x) = u_e(x)$. Therefore, $u_e(x)$ must satisfy the ordinary differential equation

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0,$$

where

$$\left. \frac{du_e}{dx} \right|_{x=0} = \left. \frac{du_e}{dx} \right|_{x=L} = 0.$$

If $f(u_e) = au_e$ and $a \neq 0$, then it can be shown that the only solution that is not identically zero occurs when $a = \mu n^2 \pi^2 / L^2$ for some integer n . Give a *brief* explanation why it is unlikely to see a solution that is not identically zero in nature. (10 points)

Solution. It is highly unlikely that the exact constant $a = \mu n^2 \pi^2 / L^2$ for some integer n will occur in nature.