Math 19. **Name________________**

Mathematical Modeling
Exam II
Fall 2002
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IMPORTANT—PLEASE READ!!! **Answer each of the following questions. For full credit be sure to show all of your work. It is sometimes possible to get the correct answer by the wrong method. In such cases very little, if any, credit will be given. All answers must appear in simplified form. Write neatly!—unreadable answers will receive little if any credit.**

Do not write in this space.

Problem Number	Possible Points	Score
1	10	
2	15	
3	20	
4	10	
5	10	
Total	65	

1. For each of the following cases, indicate whether modeling with an advection equation, a diffusion equation, or Laplace's equation is more appropriate. (10 points))

   (a) The concentration of a drug in the bloodstream after it is injected into the arm.

   (b) The distribution of a certain protein in a cell after the protein has reached a steady-state.

   (c) The spread of airborne radioactivity after Chernobyl disaster in 1986.

   (d) The spread of the oil spill from the Exxon Valdez in Prince William Sound in 1989.

   (e) The spread of killer bees when they first escaped from captivity in Brazil.

2. Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

(15 points)

(a) Verify that $u(t, x) = e^{-t} \sin x$ is a solution to (1).

(b) Verify that $u(t, x) = e^{-4t} \sin 2x$ is a solution to (1).

(c) Use these functions and the Principle of Superposition to construct a solution to (1) that satisfies the boundary and initial conditions

$$\begin{aligned} u(0, x) &= 20 \sin x + 10 \sin 2x, \\ u(t, 0) &= 0. \end{aligned}$$

3. Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u. \tag{2}$$

(20 points)

(a) Use the separation of variables technique to find all nonzero solutions to (2) subject to the conditions for all t and for $0 \leq x \leq L$ such that $u(t, 0) = u(t, L) = 0$. [*Hint:* Look for solutions of the form $u(t, x) = A(t)B(x)$.]

(b) Find the minimum positive number L with the following property: There is a solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u$$

for all t and for $0 \leq x \leq L$ such that $u(t, 0) = u(t, L) = 0$ and such that grows in size as $t \to \infty$.

4. Consider the equilibrium solution $u_e = -1$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 - u$$

satisfying the boundary conditions

$$\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, L) = 0.$$

Determine the stability of this solution. (10 points)

5. Let

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u),$$

where

$$\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, L) = 0.$$

If u has no time derivative, then u is a function only of x, i.e, $u(t, x) = u_e(x)$. Therefore, $u_e(x)$ must satisfy the ordinary differential equation

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0,$$

where

$$\left.\frac{du_e}{dx}\right|_{x=0} = \left.\frac{du_e}{dx}\right|_{x=L} = 0.$$

If $f(u_e) = a u_e$ and $a \neq 0$, then it can be shown that the only solution that is not identically zero occurs When $a = \mu n^2 \pi^2 / L^2$ for some integer n. Give a *brief* explanation why it is unlikely to see a solution that is not identically zero in nature. (10 points)