Math 19. Mathematical Modeling.
Solutions to Exam II

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1. (5 points) In each of the following cases, indicate which single model would be *most* appropriate: advection, diffusion, or a model using Laplace's equation. No justification is necessary.

   (a) The distribution of a certain protein in a cell after the protein has reached a steady-state.

   Solution. Laplace's equation.

   (b) The spread of airborne radioactivity after Chernobyl disaster in 1986.

   Solution. Advection equation.

   (c) The spread of the oil spill from the Exxon Valdez in Prince William Sound in 1989.

   Solution. Advection equation.

   (d) The spread of an anthrax epidemic in livestock.

   Solution. Diffusion equation.

   (e) The concentration of a drug in the bloodstream after it is injected into the arm.

   Solution. Advection equation.

2. (10 points) Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

   where $0 \leq x \leq \pi$ and $t \geq 0$. It is easy to verify that $u_1(t,x) = e^{-t} \sin x$ and $u_2(t,x) = e^{-4t} \sin 2x$ are solutions to (1).

   (a) Show that $u_3(t,x) = e^{-9t} \sin 3x$ is also a solution to (1).

   Solution.

$$\frac{\partial u}{\partial t} = -9e^{-9t} \sin 3x = \frac{\partial^2 u}{\partial x^2}.$$

   (b) Use the Principle of Superposition and the functions u_1, u_2, and u_3 to construct a solution to (1) that satisfies the boundary and initial conditions

$$\begin{aligned} u(0,x) &= 60 \sin x + 5 \sin 2x - 20 \sin 3x, \\ u(t,0) &= u(t,\pi) = 0. \end{aligned}$$

   Solution. Using the Principle of Superposition,

$$\begin{aligned} u(t,x) &= \alpha u_1 + \beta u_2 + \gamma u_3 \\ &= \alpha e^{-t} \sin x + \beta e^{-4t} \sin 2x + \gamma e^{-9t} \sin 3x \end{aligned}$$

is a solution to (1). Since

$$\begin{aligned} u(0,x) &= \alpha \sin x + \beta \sin 2x + \gamma \sin 3x \\ &= 60 \sin x + 5 \sin 2x - 20 \sin 3x, \end{aligned}$$

we know that $\alpha = 60$, $\beta = 5$, and $\gamma = -20$. Therefore, the solution that we seek is

$$u(t,x) = 60e^{-t} \sin x + 5e^{-4t} \sin 2x - 20e^{-9t} \sin 3x.$$

3. (15 points) Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7u.$$

(a) Use the separation of variables technique to find nonzero solutions to this equation subject to the conditions for all t and for $0 \leq x \leq L$ such that $u(t,0) = u(t,L) = 0$.

Solution. If we let $u(t,x) = A(t)B(x)$, then the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7u.$$

becomes

$$BA' = AB'' + 7AB.$$

Separating the variables,

$$\frac{A'}{A} = \frac{B''}{B} + 7 = \lambda.$$

The solution to the equation $A' = \lambda A$ is $A(t) = A(0)e^{\lambda t}$.
To solve the second equation, let

$$\frac{1}{B}B'' = \lambda - 7 = c.$$

We have three cases.

- $c = 0$. Solutions to the equation $B'' = 0$ are $B(x) = \alpha + \beta x$. Applying the boundary conditions,

  $$0 = B(0) = \alpha + \beta \cdot 0 = \alpha,$$

  and

  $$0 = B(L) = \beta \cdot L.$$

  Thus, $\beta = 0$, and there are no nonzero solutions.
- $c > 0$. The solutions to $B'' = cB$ are

  $$B(x) = \alpha e^{\sqrt{c}\,x} + \beta e^{-\sqrt{c}\,x}.$$

  Applying the boundary condition $B(0) = 0$, we obtain $0 = B(0) = \alpha + \beta$. Therefore, $\beta = -\alpha$, and

  $$B(x) = \alpha(e^{\sqrt{c}\,x} - e^{-\sqrt{c}\,x}).$$

  Applying the second boundary condition, $B(L) = 0$,

  $$0 = B(L) = \alpha(e^{\sqrt{c}\,L} - e^{-\sqrt{c}\,L}).$$

  Since the second factor can never be zero for $L > 0$, we know that $\alpha = 0$. Hence, β is also zero, and there are no nonzero solutions.

- $c < 0$. The solutions to $B'' = cB$ are

  $$B(x) = \alpha \cos(\sqrt{-c}\, x) + \beta \sin(\sqrt{-c}\, x).$$

  Applying the boundary condition $B(0) = 0$, we find that $\alpha = 0$. Therefore, $B(x) = \beta \sin(\sqrt{-c}\, x)$. Applying the second boundary condition, $0 = B(L) = \beta \sin(\sqrt{-c}\, L)$, we can find nonzero solutions if $\sqrt{-c} = n\pi/L$ or

  $$c = \lambda - 7 = -\frac{n^2\pi^2}{L^2},$$

  where n is a positive integer.

Thus, nonzero solutions must take the form

$$u(t,x) = e^{\lambda t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\lambda = 7 - \frac{n^2\pi^2}{L^2}$$

and $n = 1, 2, 3, \ldots$.

(b) Find the minimum positive number L with the following property: There is a solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7u$$

for all t and for $0 \leq x \leq L$ such that $u(t,0) = u(t,L) = 0$ and such that grows in size as $t \to \infty$.

Solution. For the solution

$$u(t,x) = e^{\lambda t} \sin\left(\frac{n\pi x}{L}\right),$$

to grow in size as $t \to \infty$,

$$\lambda = 7 - \frac{n^2\pi^2}{L^2} > 0$$

or

$$L > \frac{n\pi}{\sqrt{7}}.$$

The minimum occurs when $n = 1$ or

$$L > \frac{\pi}{\sqrt{7}}.$$

4. (10 points) Consider the equilibrium solution $u_e = 3$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 - 3u$$

satisfying the boundary conditions

$$\frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,L) = 0.$$

Determine the stability of this solution.

Solution. Since $f(u) = u^2 - 3u$ and $f'(u) = 2u - 3$. Thus, $f'(u_e) = f'(3) = 3$. To show that this is a stable solution, we must show that there exists no $\lambda > 0$ and $g \not\equiv 0$ satisfying

$$\lambda g = \frac{d^2 g}{dx^2} + 3g \text{ or } \frac{d^2 g}{dx^2} = (\lambda - 3) g.$$

If we let $c = \lambda - 3$, this is equivalent to examining the equation

$$\frac{d^2 g}{dx^2} = cg.$$

We will show that $u_e = 3$ is a (linearly) unstable equilibrium solution by finding a nonzero solution to this equation that satisfies the boundary conditions, where $\lambda > 0$. Let us assume that $c = \lambda - 3 < 0$. Then

$$\begin{aligned} g(x) &= \alpha \cos(\sqrt{-c}\, x) + \beta \sin(\sqrt{-c}\, x), \\ g'(x) &= \sqrt{-c} \cdot [-\alpha \sin(\sqrt{-c}\, x) + \beta \cos(\sqrt{-c}\, x)]. \end{aligned}$$

Applying the boundary condition, we have $0 = g'(0) = \sqrt{-c}\, \beta$. Thus, $\beta = 0$ and

$$g'(x) = \sqrt{-c} \cdot [-\alpha \sin(\sqrt{-c}\, x)].$$

Applying the second boundary condition

$$0 = g'(L) = \sqrt{-c} \cdot [-\alpha \sin(\sqrt{-c}\, L).$$

Therefore, we have a nonzero solution if $\sqrt{-c}\, L = n\pi$, where n is any positive integer. Consequently, we have found a pair (g, λ), where $0 < \lambda < 3$.

5. (10 points) Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u,$$

where $0 \leq x \leq 1$ and $u(t, 0) = u(t, 1) = 0$. Is it possible to find a solution subject to these boundary equations that grows in size as $t \to \infty$?

Solution. Substituting $u(t, x) = A(t)B(x)$ into

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u,$$

we obtain

$$\frac{1}{A} A' = \frac{1}{B} B'' + 5 = \lambda.$$

The equation $A' = \lambda A$ has solution $A(t) = A(0)e^{\lambda t}$. Thus,

$$u(t, x) = A(0)e^{\lambda t} B(x)$$

will grow as $t \to \infty$ if and only if $\lambda > 0$.

Now consider the equation $B'' = (\lambda - 5)B = cB$ with boundary conditions $B(0) = B(1) = 0$. We have three cases to consider.

- If $c = 0$, then $B(x) = \alpha + \beta x$. The boundary conditions tell us that there are no nonzero solutions.
- If $c > 0$, then

$$B(x) = \alpha e^{\sqrt{c}\, x} + \beta e^{-\sqrt{c}\, x}.$$

  Applying the boundary condition $B(0) = 0$, we find that $\beta = -\alpha$ and

$$B(x) = \alpha(e^{\sqrt{c}\, x} - e^{-\sqrt{c}\, x}).$$

  Applying the boundary condition $B(1) = 0$,

$$0 = B(1) = \alpha(e^{\sqrt{c}} - e^{-\sqrt{c}}).$$

  and $\alpha = \beta = 0$. Therefore, there are no nonzero solutions.

- If $c < 0$, then
$$B(x) = \alpha \cos(\sqrt{-c}\,x) + \beta \sin(\sqrt{-c}\,x).$$
The boundary condition $B(0) = 0$ implies $\alpha = 0$. Applying the second boundary condition,
$$0 = B(1) = \beta \sin \sqrt{-c}.$$
Thus, we have nonzero solutions if $\sqrt{-c} = n\pi$ for $n = 1, 2, 3, \ldots$ or
$$c = \lambda - 5 = -n^2\pi^2.$$
Since $\lambda = 5 - n^2\pi^2 < 0$ for every positive integer n, there does not exist a solution that grows in size as $t \to \infty$.

6. (15 points) Consider the system
$$\begin{aligned} \frac{dx}{dt} &= x - 2y - (x^2 + y^2)x \\ \frac{dy}{dt} &= 2x + y - (x^2 + y^2)y. \end{aligned}$$

(a) Show that the origin is an equilibrium point of the system.

Solution. The nullclines of the system occur when
$$\begin{aligned} \frac{dx}{dt} &= x - 2y - (x^2 + y^2)x = 0, \\ \frac{dy}{dt} &= 2x + y - (x^2 + y^2)y = 0. \end{aligned}$$
Since $(x, y) = (0, 0)$ is a solution to both equations, it must be an equilibrium solution.

(b) Prove that zero is an repelling equilibrium point.

Solution. Substituting $(0, 0)$ into the matrix
$$\begin{pmatrix} \frac{\partial}{\partial x}(x - 2y - (x^2 + y^2)x) & \frac{\partial}{\partial y}(x - 2y - (x^2 + y^2)x) \\ \frac{\partial}{\partial x}(2x + y - (x^2 + y^2)y) & \frac{\partial}{\partial y}(2x + y - (x^2 + y^2)y) \end{pmatrix} = \begin{pmatrix} 1 - 3x^2 - y^2 & -2 - 2xy \\ 2 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix},$$
we obtain the matrix
$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$
Since $\operatorname{tr}(A) = 2 > 0$ and $\det(A) = 5 > 0$, the equilibrium point $(0, 0)$ is repelling.

(c) Show that the square
$$R = \{(x, y) : -2 \le x \le 2, -2 \le y \le 2\}.$$
is a basin of attraction for the system
$$\begin{aligned} \frac{dx}{dt} &= x - 2y - (x^2 + y^2)x \\ \frac{dy}{dt} &= 2x + y - (x^2 + y^2)y. \end{aligned}$$
Hence, we can determine that the system has a period solution inside R by the Poincaré-Bendixson Theorem.

Solution. We must check the sign of the appropriate derivative on each edge of the square.

- $x = 2$ and $-2 \leq y \leq 2$. We must show that $dx/dt \leq 0$; however,

$$\frac{dx}{dt} = 2 - 2y - 2(4 + y^2) = -6 - 2y - 2y^2$$

on this edge, and this function has a maximum value of $-11/2$ at $y = -1/2$.
- $x = -2$ and $-2 \leq y \leq 2$. We must show that $dx/dt \geq 0$; however,

$$\frac{dx}{dt} = -2 - 2y + 2(4 + y^2) = 6 - 2y + 2y^2$$

on this edge, and this function has a minimum value of $11/2$ at $y = 1/2$.
- $y = 2$ and $-2 \leq x \leq 2$. We must show that $dy/dt \leq 0$; however,

$$\frac{dy}{dt} = 2x + 2 - 2(x^2 + 4) = -6 + 2x - 2x^2$$

on this edge, and this function has a maximum value of $-11/2$ at $x = 1/2$.
- $y = -2$ and $-2 \leq x \leq 2$. We must show that $dy/dt \geq 0$; however,

$$\frac{dy}{dt} = 2x - 2 + 2(x^2 + 4) = 6 + 2x + 2x^2$$

on this edge, and this function has a minimum value of $11/2$ at $x = -1/2$.

7. (5 points) Given the differential equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} - 6 \frac{\partial u}{\partial x} + u^3,$$

make the traveling wave substitution $u(t, x) = f(x - ct)$, where $c > 0$ is a constant to derive a differential equation in one variable for the function f.

Solution. Differentiating $u(t, x) = f(x - ct)$, we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= -cf'(x - ct), \\ \frac{\partial u}{\partial x} &= f'(x - ct), \\ \frac{\partial^2 u}{\partial x^2} &= f''(x - ct). \end{aligned}$$

If we let $s = x - ct$ and substitute into the equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} - 6 \frac{\partial u}{\partial x} + u^3,$$

we obtain

$$-c\frac{df}{ds} = \mu \frac{d^2 f}{ds^2} - 6 \frac{df}{ds} + f^3.$$

8. (5 points) Given the second order differential equation

$$\frac{d^2 x}{dt^2} + 3\frac{dx}{dt} + 7x(1 - x) = 0,$$

write an equivalent first-order system of differential equations.

Solution. If we let $y = dx/dt$, then

$$\frac{d^2 x}{dt^2} + 3\frac{dx}{dt} + 7x(1 - x) = 0$$

becomes

$$\frac{dy}{dt} + 3y + 7x(1-x) = 0.$$

Therefore, the system that we seek is

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -3y - 7x(1-x).\end{aligned}$$

9. (10 points) Given the equation

$$\lambda g = \frac{d^2 g}{dx^2} - (\cos x) g,$$

use the Maximum Principle to determine if there is a positive number λ and a solution g defined on the interval $0 \le x \le 1$ that vanishes at $x = 0$ and $x = 1$ and is not identically zero. In particular, write *no* and justify your answer using the Maximum Principle or explain why the Maximum Principle cannot be applied.

Solution. We will apply the Maximum Principle to show that we cannot find a pair (λ, g) satisfying the equation

$$\lambda g = \frac{d^2 g}{dx^2} - (\cos x) g,$$

First, if g has a positive maximum, then $g > 0$ and $d^2g/dx^2 < 0$ at this point. Since $\cos x > 0$ on the interval $0 \le x \le 1$, the right side of the equation must be negative. Therefore, $\lambda < 0$.

Now assume that g has no positive maximum. Then g must have a negative minimum. Thus, $g < 0$ and $d^2g/dx^2 > 0$ at this point. Therefore, the right side of the equation must be positive. Hence, $\lambda < 0$, and the Maximum Principle applies in this case.