Chapter 19 Key

(3b) $f(u) = r_1 u - r_2 u^2$; $r_1, r_2 > 0$

(4) $\mu \frac{\partial^2 u_e}{\partial x^2} + f(u_e) = 1$

(5) $\frac{\partial}{\partial x} u_e = 0$ @ $x = 0$ & L

stable if No pair $[g, \lambda]$ s.t. $\lambda \geq 0$
& $\lambda g = \mu g_{xx} + \frac{\partial f}{\partial u}(u_e(x))g$
& $g_x = 0$ @ $x = 0$ & L

1. $r_1 = r_2 = 1$ $\quad u_e = 1$ stable?

$f(u) = u - u^2$
$f'(u) = 1 - 2u$
$f'(u_e) = 1 - 2 = -1$

so: $\lambda g = \mu g_{xx} - g$
$g_{xx} = \frac{\lambda + 1}{\mu} g$

let $\frac{\lambda+1}{\mu} = c$ $\quad \mu > 0$

if $c < 0$ $\quad \lambda + 1 \leq 0$
$\lambda \leq -1$ No! λ must be ≥ 0

if $c = 0$ $\quad \lambda + 1 = 0$ $\quad \lambda = -1$ No!

if $c > 0$ $\quad \lambda + 1 > 0$ $\quad \lambda > -1$ okay.

so. $g(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$
$g'(x) = \sqrt{c}(\alpha e^{\sqrt{c}x} - \beta e^{-\sqrt{c}x})$
$g'(0) = \sqrt{c}(\alpha - \beta) = 0$ $\quad \alpha = \beta$
$g'(L) = \sqrt{c}\alpha(e^{\sqrt{c}L} - e^{-\sqrt{c}L})$
$\therefore \alpha = 0 = \beta$

$\boxed{\text{stable! No } [g, \lambda] \text{ exist.}}$

2. $r_1 = r_2 = 1$ $\quad u_e = 0$

$f(u) = u - u^2$
$f'(u) = 1 - 2u$
$f'(u_e) = 1$

so: $\lambda g = \mu g_{xx} + g$
$g_{xx} = \frac{\lambda - 1}{\mu} g$

let $\frac{\lambda - 1}{\mu} = c$ $\quad \mu > 0$

if $c < 0$ $\quad \lambda - 1 < 0$
$\lambda < 1$ okay

if $c = 0$ $\quad \lambda - 1 = 0$
$\lambda = 1$ okay

if $c > 0$ $\quad \lambda - 1 > 0$
$\lambda > 1$ okay

must test all cases until solution is found

2. continued

if $c=0$ $\lambda - 1 = 0$ $\lambda = 1$

$g(x) = \alpha + \beta x$

$g'(x) = \beta$

$g'(0) = \beta = 0$

$g'(L) = 0$

$g(x) = \alpha = \mathbb{R}$

$\boxed{\text{so } [\lambda, g] = [1, \alpha] \quad \therefore U_e = 0 \text{ is unstable}}$

3. $r_1 = 1$ $r_2 = -1$ $U_e = -1$

$F(u) = u + u^2$

$f'(u) = 1 + 2u$

$f'(U_e) = 1 - 2 = -1$

so: $g\lambda = g_{xx}\mu - g$

$g_{xx} = \frac{\lambda + 1}{\mu} g$ Now same as #1 $\therefore$

4. $r_1 = 1,\ r_2 = -1,\ u_e = 0$

$f(x) = u + u^2$

$f'(x) = 1 + 2u$

$f'(x) = 1$

so $\lambda g = g_{xx}\mu + g$ ∴ unstable as in #2.

9. $u(t,x)$ solves $\frac{\partial u}{\partial t} = 2\frac{\partial^2}{\partial x^2}u + ru$

(a) $\frac{\partial u}{\partial t} = 0$ $u = 0$ @ $x = 0, 10$

$2\frac{\partial^2}{\partial x^2}u + ru = 0$

$u_{xx} = \frac{-r}{2}u = cu$

$c > 0$ $u(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$

$u(0) = \alpha + \beta = 0$

$\alpha = -\beta$

$u(10) = \alpha(e^{\sqrt{c}10} - e^{-\sqrt{c}10}) = 0$

$\alpha = 0 = \beta$

$c = 0$ $u(x) = \alpha + \beta x$

$u(0) = \alpha = 0$

$u(10) = \beta(10) = 0$

$\beta = 0$

$c < 0$ $u(x) = \alpha\cos(\sqrt{c}x) + \beta\sin(\sqrt{c}x)$

$u(0) = \alpha = 0$

$u(10) = \beta\sin(\sqrt{c}\,10) = 0$ either $\beta = 0$ or $\beta = \mathbb{R}$ + $\sqrt{c}10 = n\pi$

$\sqrt{c} = \frac{n\pi}{10}$

so. $u(x) = \mathbb{R}\sin\left(\frac{n\pi}{10}x\right)$	or $c = -\left(\frac{n\pi}{10}\right)^2 \Rightarrow r = \frac{n^2\pi^2}{50}$

b. = same as a.

(c) $\frac{\partial u}{\partial t} = 0$ $u = 0$ @ $x = 0,\ x = 20$

$c \geq 0$ get trivial solution as above in part a.

$c < 0$ $u(x) = \alpha\cos(\sqrt{c}x) + \beta\sin(\sqrt{c}x)$

$u(0) = \alpha = 0$

$u(20) = \beta\sin(\sqrt{c}\,20)$ $\beta = 0$ or $\beta = \mathbb{R}$ + $\sqrt{c}20 = n\pi \Rightarrow c = -\left(\frac{n\pi}{20}\right)^2$

$\boxed{r = \frac{n^2\pi^2}{200}}$