Chapter 17 Key

1. (a) I $h(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$

$h(0) = \alpha + \beta = 0$

$\alpha = -\beta$

$h(R) = \alpha(e^{\sqrt{c}R} - e^{-\sqrt{c}R}) = 0$

$\boxed{\alpha = 0 = \beta}$

II $h(x) = \alpha + \beta x$

$h(0) = \alpha = 0$

$h(R) = \beta R = 0$

$\boxed{\beta = \alpha = 0}$

III $h(x) = \alpha\cos(\sqrt{c}x) + \beta\sin(\sqrt{c}x)$

$h(0) = \alpha = 0$

$h(R) = \beta\sin(\sqrt{c}R) = 0$

$\boxed{\begin{array}{l}\alpha = 0 \\ \beta = 0 \quad \text{or} \quad c = -\left(\frac{n\pi}{R}\right)^2 \\ \qquad\qquad\quad \beta = \mathbb{R}\end{array}}$

(c) I. $h(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$

$h'(x) = \sqrt{c}(\alpha e^{\sqrt{c}x} - \beta e^{\sqrt{c}x})$

$h'(0) = \sqrt{c}(\alpha - \beta) = 0$

$\alpha = \beta$

$h(R) \quad \alpha(e^{\sqrt{c}R} + e^{-\sqrt{c}R}) = 0$

$\boxed{\alpha = 0 = \beta}$

II. $h(x) = \alpha + \beta x$

$h'(x) = \beta$

$h'(0) = \beta = 0$

$h(R) = \alpha = 0$

$\boxed{\alpha = \beta = 0}$

III. $h(x) = \alpha\cos(\sqrt{c}x) + \beta\sin(\sqrt{c}x)$

$h'(x) = \sqrt{c}\,(-\alpha\sin(\sqrt{c}x) + \beta\cos(\sqrt{c}x))$

$h'(0) = \sqrt{c}\,\beta = 0 \qquad \beta = 0$

$h(R) = \alpha\cos(\sqrt{c}R) = 0$

$\boxed{\begin{array}{l}\beta = 0 \\ \alpha = 0 \quad \text{or} \quad c = -\left(\frac{(n+\frac{1}{2})\pi}{R}\right)^2\end{array}}$

1.(e) I. $h(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$

$h(0) = \alpha + \beta = 0$

$\alpha = -\beta$

$h(R) = \alpha \left(e^{\sqrt{c}R} + e^{-\sqrt{c}R}\right) = 1$

$$\boxed{\begin{aligned} \alpha &= \frac{1}{e^{\sqrt{c}R} + e^{-\sqrt{c}R}} \\ \beta &= -\left(e^{\sqrt{c}R} + e^{-\sqrt{c}R}\right)^{-1} \end{aligned}}$$

II. $h(x) = \alpha + \beta(x)$

$h(0) = \alpha = 0$

$h(R) = \beta R = 1$

$$\boxed{\alpha = 0,\ \beta = 1/R}$$

III. $h(x) = \alpha\cos(\sqrt{c}x) + \beta\sin(\sqrt{c}x)$

$h(0) = \alpha = 0$

$h(R) = \beta\sin(\sqrt{c}R) = 1$

$$\boxed{\alpha = 0 \quad \beta = \left(\sin(\sqrt{c}R)\right)^{-1}}$$

(f) I. $h(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$

$h(0) = \alpha + \beta = -1$

$\alpha = -1 - \beta$

$h(R) = -(1+\beta)e^{\sqrt{c}R} + \beta e^{-\sqrt{c}R} = 1$

$$\boxed{\begin{aligned} \beta &= \left(1 + e^{\sqrt{c}R}\right) / \left(-e^{\sqrt{c}R} + e^{-\sqrt{c}R}\right) \\ \alpha &= -1 - \left(1 + e^{\sqrt{c}R}\right) \div \left(-e^{\sqrt{c}R} + e^{-\sqrt{c}R}\right) \end{aligned}}$$

II. $h(x) = \alpha + \beta x$

$h(0) = \alpha = -1$

$h(R) = -1 + \beta R = 1$

$\beta = 2/R$

$$\boxed{\alpha = -1,\ \beta = 2/R}$$

III. $h(x) = \alpha\cos(-\sqrt{c}x) + \beta\sin(-\sqrt{c}x)$

$h(0) = \alpha = -1$

$h(R) = -\cos(-\sqrt{c}R) + \beta\sin(-\sqrt{c}R) = 1$

$$\boxed{\alpha = -1,\ \beta = \frac{1 + \cos(-\sqrt{c}R)}{\sin(-\sqrt{c}R)}}$$

2. $\frac{\partial u}{\partial t} = 2\frac{\partial u}{\partial x^2}$

$A'B = 2B''A$

$A'/A = 2B''/B = \lambda \Rightarrow \quad A = A_0 e^{\lambda t}$

for B $\qquad B'' = \frac{\lambda}{2}B$

(a) I. $h(x) = \alpha e^{\sqrt{\lambda/2}x} + \beta e^{-\sqrt{\lambda/2}x}$

$h' = \sqrt{\lambda/2}\left(\alpha e^{\sqrt{\lambda/2}x} - \beta e^{-\sqrt{\lambda/2}x}\right)$

$h'(0) = \sqrt{\lambda/2}\,(\alpha - \beta) = 0$

$\alpha = \beta$

$h'(10) = \sqrt{\lambda/2}\,\alpha\left(e^{\sqrt{\lambda/2}\,10} - e^{-\sqrt{\lambda/2}\,10}\right) = 0$

$\alpha = 0 = \beta$

II. $h(x) = \alpha + \beta x$

$h'(x) = \beta$

$h'(0) = \beta = 0$

$\alpha = \mathbb{R}$

$u(t,x) = A_0 e^{\lambda t}\alpha$

III $h(x) = \alpha\cos(\sqrt{-\lambda/2}\,x) + \beta(\sin\sqrt{-\lambda/2}\,x)$

$h'(x) = \sqrt{-\lambda/2}\left(-\alpha\sin\sqrt{-\lambda/2}\,x + \beta\cos\sqrt{-\lambda/2}\,x\right)$

$h'(0) = \sqrt{-\lambda/2}\,(\beta) = 0 \qquad \beta = 0$

$h'(10) = \sqrt{-\lambda/2}\left(-\alpha\sin(\sqrt{-\lambda/2}\,10)\right) = 0$

$-\alpha = \mathbb{R} \qquad \lambda = -2\left(\frac{n\pi}{10}\right)^2$

$u(t,x) = A_0 e^{\lambda t}\left(\alpha\cos\left(\frac{n\pi}{10}x\right)\right)$

(b) I.

$h'(0) = 0 \quad \therefore \quad \alpha = \beta$

$h(10) = \alpha\left(e^{\sqrt{\lambda/2}\,10} + e^{-\sqrt{\lambda/2}\,10}\right) = 0$

$\alpha = 0 = \beta$

II.

$h'(0) = 0 \quad \therefore \quad \beta = 0$

$h(10) = \alpha = 0$

III

$h'(0) = 0 \;;\; B = 0$

$h(10) = \alpha\cos\left(\sqrt{-\lambda/2}\,10\right) = 0$

$\alpha = \mathbb{R}$ if $\sqrt{-\lambda/2} = \frac{(n+1/2)\pi}{10}$

$u(t,x) = A_0 e^{\lambda t}\left(\alpha\cos\left(\left(\frac{n+1/2}{10}\right)\pi x\right)\right)$

I

C. $h(0) = \alpha + \beta = 0$

$\alpha = -\beta$

$h'(10) = \sqrt{\lambda/2}\,\alpha\left(e^{\sqrt{\lambda/2}\,10} - e^{-\sqrt{\lambda/2}\,10}\right)$

$\alpha = 0 = \beta$

II $h(0) = \alpha + \beta(0) = 0$

$\alpha = 0$

$h'(10) = (10)\beta = 0$

$\beta = 0$

III $h(0) = \alpha = 0$

$h'(10) = \sqrt{\lambda/2}\,\beta\cos\sqrt{\lambda/2}\,(10) = 0$

$\beta = \mathbb{R}$ in $\sqrt{\lambda/2} = \frac{(n+1/2)\pi}{10}$

$u(t,x) = A_0 e^{\lambda t}\left(\beta \sin\left(\frac{n+1/2}{10}\pi x\right)\right)$

3. $u_t = -cu_x + ru$ $\quad u = AB;\ u_t = A'B;\ u_x = AB'$

$A'B = -cAB' + rAB$

$A'/A = -cB'/B + r = \lambda$ $\quad A = A_0 e^{\lambda t}$

$B' = -\left(\frac{\lambda - r}{c}\right)B = \frac{r-\lambda}{c}B = \gamma B$

$B = B_0 e^{\left(\frac{r-\lambda}{c}\right)*}$

$u(t,x) = AB = A_0 e^{\lambda t} B_0 e^{\left(\frac{r-\lambda}{c}\right)*}$