Math 19. Mathematical Modeling
Reviewing for Midterm I

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1. Go over the extra problems and solutions for Chapters 1–12 (pp. 481–488 in the textbook).

2. Be able to model a problem with a differential equation. For example:

   The rhinoceros is now extremely rare. Suppose that enough game preserve land is set aside so that there is sufficient room for many more rhinoceros territories than there are rhinoceros. Consequently, there will be no danger of overcrowding. However, if the population is too small, fertile adults have difficulty finding each other when it is time to mate. Write a differential equation that models the rhinoceros population based on these assumptions. (Note that there is more than one reasonable model that fits these assumptions).

3. Understand and be able to apply the exponential growth, logistic growth, and the predator-prey models to different situations.

4. Be able to analyze a differential equation of the form $dx/dt = f(x)$ given information or an explicit formula for $f(x)$. For example:

   - Consider the following differential equation for the function $x(t)$:

     $$\frac{dx}{dt} = x(x+1)(x-1)$$

     (a) What are the equilibrium points?
     (b) Which equilibrium points are stable?
     (c) If $x(0) = 2$, what happens as t gets very large?
     (d) If $x(0) = -2$, what happens as t gets very large?

   - Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information.

- The only equilibrium points in the population are $P = 0$, $P = 10$, and $P = 50$.
- If the population is 100, the population decreases.
- If the population is 25, the population increases.

(a) Sketch the possible phase lines for this system for $P > 0$. (There are two.)

(b) Give a rough sketch of the corresponding function $f(P)$ for each of your phase lines.

(c) Give a formula for the functions $f(P)$ whose graph agrees (qualitatively) with the rough sketches in part (2) for each of your phase lines.

5. Be able to do phase plane analysis for linear systems. For example:

   Consider the system of differential equations

$$\frac{dx}{dt} = x - y,$$
$$\frac{dy}{dt} = y + 2x.$$

   (a) Draw and label the x and y null clines.

   (b) Label the equilibrium point and decide if it is stable. Justify your answer.

   (c) On the drawing in part (a), label the regions where $dx/dt > 0$ and where $dx/dt < 0$. Do the same for dy/dt.

6. Be able to do phase plane analysis for nonlinear systems. For example:

   Consider the system of differential equations

$$\frac{dx}{dt} = x^2 - y^2 + 1,$$
$$\frac{dy}{dt} = -x.$$

   (a) Draw and label the x and y null clines.

   (b) Find the equilibrium points and label them on your drawing from part (a).

   (c) Decide if the equilibrium points are stable. Justify your answer.

   (d) On the drawing in part (a), label the regions where $dx/dt > 0$ and where $dx/dt < 0$. Do the same for dy/dt.

   (e) What happens to

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

   for small but positive t if $x(0) = 1$ and $y(0) = \sqrt{2}$?

7. Understand the chain rule for functions in two variables and be able to compute partial derivatives.