Math 19. Mathematical Modeling
Phase Plane Analyis

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The system of equations

$$\begin{aligned} \frac{dx}{dt} &= f(x; y) \\ \frac{dy}{dt} &= g(x; y) \end{aligned} \qquad (1)$$

is completely predictive. If you choose a starting point in the xy-plane, then there is exactly one solution to the system that starts at your chosen point.

- For any starting point in the xy-plane, there is a unique solution to (1).
- Think of $v(t)$ as tracing out a trajectory in the xy-plane as t increases. The goal is to predict the behavior of this trajectory.
- The phase plane analysis is done to help predict the trajectory.
- **Step 1.** Draw the curves where $f(x; y) = 0$. These curves are called the x **null clines**. When $v(t)$ lies on one of these curves, $dx/dt = 0$. Draw vertical slash marks on the x null clines to remind yourself that a trajectory crossing the null cline can only do so if it is moving in a vertical direction at the instant of crossing.
- **Step 2.** Draw the curves where $y(x; y) = 0$. These curves are called the y **null clines**. When $v(t)$ lies on one of these curves, $dy/dt = 0$.

Draw horizontal slash marks on the y null clines to remind yourself that a trajectory crossing the null cline can only do so if it is moving in a horizontal direction at the instant of crossing.

² Step 3. Label the points where the x and y null clines intersect. These intersections are called **equilibrium points**. If v(t) is ever at one of these points , then both dx=dt and dy=dt vanish. This means that the trajectory stays at the point for all time. If the system described by (1) is going to settle into a steady state, then v(t) will approach on of the equilibrium points as t ! 1 .

² Step 4. Label the regions of the xy-plane where dx=dt < 0 and where dx=dy > 0. These regions are always separated by x null clines. Likewise, label the regions where dy=dt is positive and negative.

² Step 5. Go back and put arrows on the vertical hash marks of the x null clines. These arrows indicate whether the motion across the null cline is up or down. The arrows are up on the parts of the x null cline that are in the dy=dt > 0 region, and down on those parts of the x null cline in the dy=dt < 0 regions. Likewise, draw arrows on the horizontal slash marks of the y null clines. These arrows are pointing right on the parts of the y null cline in the dx=dt > 0 regions and left point on the parts in the dx=dt < 0 regions.

² Step 6.

(a) If dx=dt > 0 and dy=dt > 0, then both x(t) and y(t) are increasing and the trajectory moves up and right.

(b) If dx=dt > 0 and dy=dt < 0, the trajectory moves down and right.

(c) If dx=dt < 0 and dy=dt > 0, the trajectory moves up and left.

(d) If dx=dt < 0 and dy=dt < 0, the trajectory moves down and left.