# Math 19. Mathematical Modeling
# Phase Lines and One Component Systems*

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An ***autonomous differential equation*** is an equation of the form

$$\frac{dy}{dt} = f(y). \tag{1}$$

For each initial condition $y(0) = y_0$, this is equation will have a unique solution. We can use phase lines and direction fields to help us sketch the graph of a solution to (1).

Autonomous equations have direction fields of a very special kind. The direction field does not depend on the variable $t$. Therefore, the slope marks along any horizontal line are all parallel. (You should should convince yourself of this by doing several examples with *dfield*). For this reason the slope field of any autonomous equation contains a great deal of redundant information. If we know the slope field along a single vertical line, then we can recover the slope field for the entire plane. Instead of drawing the entire slope field, we can draw a single line, the ***phase line***, that contains all of the relavant information.

Let us analyze the equation

$$\frac{dy}{dt} = \left(1 - \frac{y}{25}\right)^2 \left(\frac{y}{8} - 1\right) y^3$$

with initial condition $y(0) = 10$.

- ***Step 1.*** Draw the $y$-line. This is the ***phase line***.

*For more on phase line analysis, see the source of this handout, the second edition of Blanchard, Devaney, and Hall's textbook, *Differential Equations*, published by Brooks/Cole.

- ***Step 2.*** Find the equilibrium points. These point occur where

$$\frac{dy}{dt} = \left(1 - \frac{y}{25}\right)^2 \left(\frac{y}{8} - 1\right) y^3$$

The equilibrium point for out example occur at $y = 0$, $y = 8$, and $y = 25$. Mark these points on the phase line.

- ***Step 3.*** Find the intervals of $y$-values for which $f(y) > 0$ and mark these intervals on the phase line with arrow pointing upward.

- ***Step 4.*** Find the intervals of $y$-values for which $f(y) < 0$ and mark these intervals on the phase line with arrow pointing downward.

- ***Step 5.*** Draw the phase line.

  - If $f(y_0) = 0$, then $y(0)$ is a equilibrium point and $y(t) = y_0$ for all values of $t$.
  - If $f(y_0) > 0$, then $y(t)$ is increasing for all $t$ and approaches either the next alrgest equilibrium point or grows infinitely large.
  - If $f(y_0) < 0$, then $y(t)$ is decreasing for all $t$ and approaches either the next smallest equilibrium point or grows infinitely small as $t \to \infty$.

- ***Step 6.*** Sketch the solution to the equation.