Math 19. Mathematical Modeling.
Solutions to Exam II

T. Judson

Fall 2002

1. For each of the following cases, indicate whether modeling with an advection equation, a diffusion equation, or Laplace's equation is more appropriate. (10 points))

   (a) The concentration of a drug in the bloodstream after it is injected into the arm.

   Solution. Advection.

   (b) The distribution of a certain protein in a cell after the protein has reached a steady-state.

   Solution. Laplace.

   (c) The spread of airborne radioactivity after Chernobyl disaster in 1986.

   Solution. Advection.

   (d) The spread of the oil spill from the Exxon Valdez in Prince William Sound in 1989.

   Solution. Advection.

   (e) The spread of killer bees when they first escaped from captivity in Brazil.

   Solution. Diffusion.

2. Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

   (15 points)

   (a) Verify that $u(t, x) = e^{-t} \sin x$ is a solution to (1).

   Solution. Differentiating u with respect to t, we get

$$\frac{\partial u}{\partial t} = -e^{-t} \sin x.$$

On the other hand, differentiating u with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} e^{-t} \sin x = e^{-t} \frac{\partial^2}{\partial x^2} \sin x = -e^{-t} \sin x.$$

Therefore, $u(t, x) = e^{-t} \sin x$ is a solution to (1).

(b) Verify that $u(t, x) = e^{-4t} \sin 2x$ is a solution to (1).

Solution. Differentiating u with respect to t, we get

$$\frac{\partial u}{\partial t} = -4e^{-4t} \sin 2x.$$

On the other hand, differentiating u with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} e^{-4t} \sin 2x = e^{-4t} \frac{\partial^2}{\partial x^2} \sin 2x = -4e^{-4t} \sin 2x.$$

Therefore, $u(t, x) = e^{-4t} \sin 2x$ is a solution to (1).

(c) Use these functions and the Principle of Superposition to construct a solution to (1) that satisfies the boundary and initial conditions

$$\begin{aligned} u(0, x) &= 20 \sin x + 10 \sin 2x, \\ u(t, 0) &= 0. \end{aligned}$$

Solution. Using the Principle of Superposition

$$u(t, x) = \alpha e^{-t} \sin x + \beta e^{-4t} \sin 2x$$

is a solution to (1) for all real numbers α and β. This solution certainly satisfies the initial condition $u(t, 0) = 0$. If we choose $\alpha = 20$ and $\beta = 10$, then $u(0, x) = 20 \sin x + 10 \sin 2x$ is satisfied.

3. Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u. \tag{2}$$

(20 points)

(a) Use the separation of variables technique to find all nonzero solutions to (2) subject to the conditions for all t and for $0 \leq x \leq L$ such that $u(t, 0) = u(t, L) = 0$. [*Hint:* Look for solutions of the form $u(t, x) = A(t)B(x)$.]

Solution. If $u(t, x) = A(t)B(x)$, then

$$B \frac{dA}{dt} = A \frac{d^2 B}{dx^2} + 5AB$$

or

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{B}\frac{d^2B}{dx^2} + 5 = \lambda$$

for some constant λ. Solving

$$\frac{dA}{dt} = \lambda A,$$

we have $A(t) = A(0)e^{\lambda t}$. For

$$\frac{1}{B}\frac{d^2B}{dx^2} + 5 = \lambda \text{ or } \frac{d^2B}{dx^2} = cB,$$

where $c = \lambda - 5$, the boundary conditions cannot be satisfied unless $c < 0$.

- If $c = 0$, then $B(x) = \alpha + \beta x$. The boundary conditions tell us that $\alpha = \beta = 0$. Therefore, we have no nonzero solutions if $c = \lambda - 5 = 0$.
- If $c > 0$, then $B(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$. The boundary conditions tell us that

$$\begin{aligned} B(0) &= \alpha + \beta = 0 \\ B(L) &= \alpha e^{\sqrt{c}L} + \beta e^{-\sqrt{c}L} = 0. \end{aligned}$$

  Therefore, $\beta = -\alpha$ and

$$\alpha(e^{\sqrt{c}L} - e^{-\sqrt{c}L}) = 0.$$

  Since $e^{\sqrt{c}L} - e^{-\sqrt{c}L} > 0$ for $L > 0$, It must be the case that $\alpha = 0$. Thus, we have no nonzero solutions if $c = \lambda - 5 > 0$.
- If $c < 0$, then

$$B(x) = \alpha \cos(\sqrt{-c}x) + \beta \sin(\sqrt{-c}x).$$

  The boundary condition $B(0) = 0$ tells us that $\alpha = 0$. The second boundary condition, $B(L) = 0$, says that

$$0 = B(L) = \beta \sin(\sqrt{-c}x).$$

  Therefore, we have a nonzero solution if $\sqrt{-c} = n\pi/L$ or $c = \lambda - 5 = -n^2\pi^2/L^2$ for any integer n. So the nonzero solutions of (2) are

$$u(t,x) = \alpha e^{\lambda t} \sin\left(\frac{n\pi x}{L}\right),$$

  where

$$\lambda = 5 - \frac{n^2\pi^2}{L^2}.$$

(b) Find the minimum positive number L with the following property: There is a solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u$$

for all t and for $0 \leq x \leq L$ such that $u(t,0) = u(t,L) = 0$ and such that grows in size as $t \to \infty$.

Solution.

$$u(t,x) = \alpha e^{\lambda t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\lambda = 5 - \frac{n^2\pi^2}{L^2}.$$

to grow without bound, it is necessary that we choose $\lambda > 0$ or $L^2 > n^2\pi^2/5$. The smallest such number occurs when $n = 1$ or $L > \pi/\sqrt{5}$.

4. Consider the equilibrium solution $u_e = -1$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 - u$$

satisfying the boundary conditions

$$\frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,L) = 0.$$

Determine the stability of this solution. (10 points)

Solution. Since $f(u) = u^2 - u$, we know that $f'(-1) = -3$. Therefore, we must determine if there is a $\lambda \geq 0$ such that

$$\lambda g = \frac{d^2 g}{dx^2} - 3g \tag{3}$$

has a solution for g that is not identically zero. If we let $c = \lambda + 3$, then this equation becomes

$$\frac{d^2 g}{dx^2} = cg$$

subject to the boundary conditions

$$\left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=L} = 0.$$

We have the usual three cases $c < 0$, $c = 0$, and $c > 0$

- If $c = \lambda + 3 < 0$, then $\lambda < -3$, and we can find no pair (λ, g).
- If $c = \lambda + 3 = 0$, then $\lambda = -3$, and we can find no pair (λ, g).

- If $c > 0$, then

  $$\begin{aligned} g(x) &= \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x} \\ \frac{dg}{dx} &= \alpha\sqrt{c}e^{\sqrt{c}x} - \beta\sqrt{c}e^{-\sqrt{c}x}. \end{aligned}$$

  The boundary condition at $x = 0$ tells us that $\alpha = \beta$. Therefore, at $x = l$, we know that

  $$0 = \left.\frac{dg}{dx}\right|_{x=L} = \alpha\sqrt{c}e^{\sqrt{c}L} - \alpha\sqrt{c}e^{-\sqrt{c}L} = \alpha\sqrt{c}(e^{\sqrt{c}L} - e^{-\sqrt{c}L}).$$

  Since $e^{\sqrt{c}L} > e^{-\sqrt{c}L}$ for $L > 0$, it must be the case that $\alpha = 0$. Therefore, the solution to (3) is $g \equiv 0$.

Since we can find no pair (λ, g), the solution $u_e = -1$ is stable.

5. Let

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} + f(u),$$

where

$$\frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,L) = 0.$$

If u has no time derivative, then u is a function only of x, i.e, $u(t,x) = u_e(x)$. Therefore, $u_e(x)$ must satisfy the ordinary differential equation

$$\mu\frac{d^2u_e}{dx^2} + f(u_e) = 0,$$

where

$$\left.\frac{du_e}{dx}\right|_{x=0} = \left.\frac{du_e}{dx}\right|_{x=L} = 0.$$

If $f(u_e) = au_e$ and $a \neq 0$, then it can be shown that the only solution that is not identically zero occurs When $a = \mu n^2\pi^2/L^2$ for some integer n. Give a *brief* explanation why it is unlikely to see a solution that is not identically zero in nature. (10 points)

Solution. It is highly unlikely that the exact constant $a = \mu n^2\pi^2/L^2$ for some integer n will occur in nature.