Math 19. Mathematical Modeling.
Solutions to Exam I

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Fall 2002

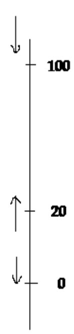
1. Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

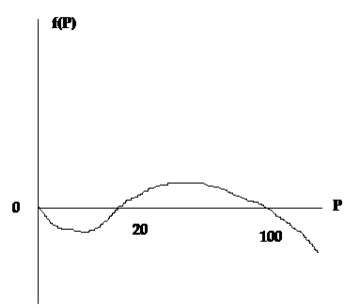
   - The population at $P = 0$ remains constant.
   - A population close to 0 will decrease.
   - A population of $P = 20$ will increase.
   - A population of $P > 100$ will decrease.

   Answer each of the following questions. (10 points)

   (a) Sketch the simplest possible phase line that agrees with the experimental information above.

   (b) Graph a rough sketch of the function $f(P)$ for the phase line of part (a).





2. Consider the following differential equation for the function $y(t)$:

$$\frac{dy}{dt} = y^2 - 6y - 16.$$

(10 points)

(a) What are the equilibrium points?

Solution. The equation

$$y^2 - 6y - 16 = (y-8)(y+2) = 0$$

has solutions $y = 8$ and $y = -2$, which are the equilibrium points.

(b) Which equilibrium points are stable?

Solution. $y = 8$ is unstable and $y = -2$ is stable.

(c) If $y(0) = 5$, what happens as t gets very large?

Solution. As $t \to \infty$, $y(t) \to -2$.

(d) If $y(0) = 10$, what happens as t gets very large?

Solution. As $t \to \infty$, $y(t) \to \infty$.

3. Given $f(x, y) = y + \cos(x^2 y)$, compute the following partial derivatives: (9 points)

(a) $\frac{\partial}{\partial x} f(x, y)$

Solution.

$$\frac{\partial}{\partial x} f(x, y) = -2xy \sin(x^2 y)$$

(b) $\frac{\partial}{\partial y} f(x, y)$

Solution.

$$\frac{\partial}{\partial y} f(x, y) = 1 - x^2 \sin(x^2 y)$$

(c) $\frac{\partial^2}{\partial x^2} f(x, y)$

Solution.

$$\frac{\partial^2}{\partial x^2} f(x, y) = -2y \sin(x^2 y) - 4x^2 y^2 \cos(x^2 y)$$

4. Consider the following two predator-prey systems of differential equations:

(i)

$$\frac{dx}{dt} = 10x\left(1-\frac{x}{10}\right) - 20xy,$$
$$\frac{dy}{dt} = -5y + \frac{xy}{20}.$$

(ii)

$$\frac{dx}{dt} = 0.3x - \frac{xy}{100},$$
$$\frac{dy}{dt} = 15y\left(1-\frac{y}{15}\right) + 25xy.$$

In one of these systems, the prey are very large animals and the predators are very small animals, such as elephants and mosquitos. Thus, it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey, such as whales and krill. Determine which system is which and provide a justification for your answer. (10 points)

Solution. (i) corresponds to the large predator-small prey situation. The term $-20xy$ means that it takes many prey to benefit one predator. (ii) corresponds to the small predator-large prey situation. The term $25xy$ means that one prey benefits many predators.

5. Consider the system of differential equations

$$\frac{dx}{dt} = x - y,$$
$$\frac{dy}{dt} = 3y - 2x.$$

(12 points)

(a) Draw and label the x and y null clines.

(b) Label the equilibrium point and decide if it is stable. Justify your answer.

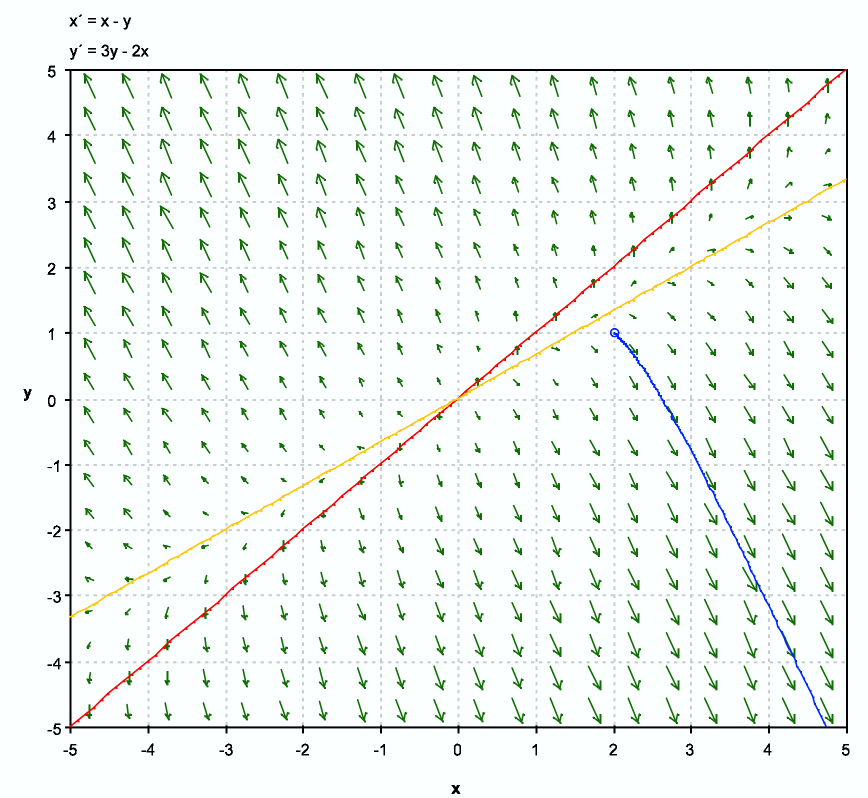
(c) On the drawing in part (a), label the regions where $dx/dt > 0$ and where $dx/dt < 0$. Do the same for dy/dt.

(d) For the initial condition $x(0) = 2$ and $y(0) = 1$, sketch the trajectory in the phase plane.

Solution. The x null cline is $y = x$ and the y null cline is $y = 3x/2$. The matrix form of this system is $d\mathbf{v}/dt = A\mathbf{v}$, where

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

Since the determinant of A is 1 and the trace of A is 4, the equilibrium point, $(0,0)$, is unstable.



6. Consider the following assumptions concerning the fraction of a piece of bread covered by mold.

   - Mold spores fall on the bread at a constant rate.

- When the proportion covered is small, the fraction of the bread covered by mold increases at a rate proportional to the amount of bread covered.
- When the fraction of the bread covered by mold is large, the growth rate decreases.
- In order to survive, mold must be in contact with the bread.

Using these assumptions, write a differential equation that models the proportion of a piece of bread covered by mold. Explain your model in one or two carefully worded sentences. *Note that there is more than one reasonable model that fits these assumptions.* (10 points)

Solution. One possible model is

$$\frac{dP}{dt} = kP(1 - P),$$

where P is the fraction of the bread covered by mold. When P is small, the growth is nearly exponential. As P approaches 1, the growth rate decreases (but is still positive).