Math 192r, Problem Set #13: Solutions

1. (a) *How many lattice paths from* $(0,0)$ *to* (m,n) *remain the same when you rotate them by 180 degrees about* $(\frac{m}{2},\frac{n}{2})$*? Prove your answer.*

   By symmetry, the lattice path must cross the line $x+y=(m+n)/2$ at the point $(m/2,n/2)$, which is impossible if m and n are both odd (since for every point on the lattice path, either the x-coordinate or the y-coordinate is an integer).

   If m and n are both even, then $(m/2,n/2)$ is a lattice point, and we can see that every lattice path from $(0,0)$ to $(m/2,n/2)$, when rotated 180 degrees about the point $(m/2,n/2)$, yields a lattice path from $(0,0)$ to (m,n) that is invariant under 180 rotation. Since it is also clear that every invariant lattice path is of this form, the number of such paths is just $\frac{(m/2+n/2)!}{(m/2)!(n/2)!}$.

   If m is even and n is odd, then $(m/2,n/2)$ is the midpoint of the segment joining $(m/2,(n-1)/2)$ and $(m/2,(n+1)/2)$, and this segment must be part of the lattice path. In particular, the lattice path must go from $(0,0)$ to $(m/2,(n-1)/2)$. In this case, the lattice paths from $(0,0)$ to (m,n) that are invariant under rotation are in bijection with the lattice paths from $(0,0)$ to $(m/2,(n-1)/2)$, and the number of paths is just $\frac{(m/2+n/2-1/2)!}{(m/2)!(n/2-1/2)!}$.

   Likewise, if m is odd and n is even, the number of invariant paths is $\frac{(m/2+n/2-1/2)!}{(m/2-1/2)!(n/2)!}$.

2. (a) *How many lattice paths from* $(0,0)$ *to* (n,n) *remain the same when you flip them across the diagonal joining* $(n,0)$ *and* $(0,n)$*? Prove your answer.*

   Such a path must cross the line $x+y=n$ at some point (i,j). If we flip the path from $(0,0)$ to (i,j) across the diagonal, we get a lattice path from $(0,0)$ to (n,n) of the specified kind, and every such path arises in this way. Thus the paths from $(0,0)$ to (n,n) that are invariant under reflection are in bijection with lattice paths from $(0,0)$ to the line $x+y=n$, of which there are $\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}=2^n$.

   (b) *What is the sum of the q-weights of these lattice paths? Conjecture an answer.*

Consider a lattice path from $(0, 0)$ to (i, j), with $i + j = n$. If its q-weight is q^m, then the associated path from $(0, 0)$ to (n, n) (obtained by reflection) has q-weight q^{2m+j^2}. (To see this, split the area under the path into three parts: the part below and to the left of (i, j), the part above and to the right of (i, j), and the part below and to the right of (i, j); these regions have area m, m, and j^2, respectively.) Therefore, using the function $P_{m,n}(q)$ from the previous problem set, we can write the sum of the q-weights of the symmetrical lattice paths from $(0, 0)$ to (n, n) as $\sum_{j=0}^{n} q^{j^2} P_{n-j,j}(q^2)$.

Here we can use Maple. First, from what we learned in class, we can write a program for $P_{m,n}$:

```
P := proc(m,n) local i,j;
        expand(simplify(mul(mul(
        (1-q^(i+j))/(1-q^(i+j-1)),
        i=1..m),j=1..n))); end;
```

Then we can write a program to q-count reflection-invariant lattice paths

```
S := proc(n) local j;
        expand(simplify(add(q^(j^2)*
        subs(q=q^2,P(n-j,j)),j=0..n)));
        end;
```

A little bit of exploration will then yield the observation that $S(n)$ divided by $S(n-1)$ equals $1 + q^{2n-1}$, so that

$$S(n) = (1 + q)(1 + q^3) \cdots (1 + q^{2n-1}).$$

In fact, we can prove this combinatorially by dividing the area under the lattice path into L-shapes with their corners at the lower right. Each L-shape, being symmetrical, contains an odd number of squares. Also, since the L-shapes fit together to form a (flipped) Young diagram, their sizes must be distinct, with the the largest possible L-shape being of size $2n - 1$. Finally, note that if we take any set of odd numbers from 1 to $2n-1$, L-shapes of those sizes may be fit together to form a flipped Young diagram that is reflection-invariant and whose boundary is a reflection-invariant lattice path.

(c) *Why is there no part (b) for question 1?*

Because all of the paths have the same q-weight, namely $q^{AB/2}$, so it would have been silly to ask the question.

(Actually, the preceding paragraph *would* have been the right answer, if someone else had asked the question, or if I had asked it in class. But since I asked the question *on the homework*, I obviously didn't think it was too silly a question to ask! A psychologically accurate answer is that I originally included a part (b), then deleted it, and then decided to re-include it, but in a slightly off-beat way that would hopefully be amusing or at least provocative.)