Mathematics 191
Approximating Binomial Coefficients by Wallis's product

This gets the key approximation of section 3.10 without Stirling's approximation. Let

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^{n-2} x(1-\cos^2 x)\,dx = \int_0^{\pi/2} \sin^{n-2} x\,dx - \int_0^{\pi/2} \cos x(\sin^{n-2} x \cos x)$$

Integrate the second term by parts:
If $n \geq 2$:

$$\int_0^{\pi/2} \cos x(\sin^{n-2} x \cos x)\,dx = \cos x \frac{\sin^{n-1} x}{n-1}]_0^{\pi/2} + \int_0^{\pi/2} \sin x \frac{sin^{n-1} x}{n-1}\,dx$$

The first term on the right-hand side is zero at both endpoints of integration. So

$$I_n = I_{n-2} - \frac{I_n}{n-1}$$

and

$$(n-1)I_n = (n-1)I_{n-2} - I_n$$

or

$$\boxed{I_{n-2} = \tfrac{n}{n-1} I_n}$$

Now

$$I_0 = \int_0^{\pi/2} dx = \pi/2$$

and

$$I_1 = \int_0^{\pi/2} \sin x\,dx = 1$$

So

$$\pi/2 = \frac{I_0}{I_1} = \frac{2/1 I_2}{3/2 I_3} = \frac{2/1 * 4/3 I_2}{3/2 * 5/4 I_3} = \frac{2/1 * 4/3 * 6/5 I_6}{3/2 * 5/4 * 7/6 I_7} = \frac{2/1 * 4/3 * 6/5 * 8/7 I_8}{3/2 * 5/4 * 7/6 I_7} = \cdots$$

For large m we can approximate $\frac{I_{2m}}{I_{2m-1}}$ by 1 and so:

$$\pi/2 \approx \frac{2/1 \cdot 4/3 \cdot 6/5 \cdots 2m/2m-1}{3/2 \cdot 5/4 \cdot 7/6 \cdots \frac{2m-1}{2m-2}} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2m-2)^2 \cdot 2m}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots (2m-1)^2}$$

and

$$\sqrt{\pi/2} \approx \frac{2 \cdot 4 \cdot 6 \cdots (2m-2)\sqrt{2m}}{3 \cdot 5 \cdot 7 \cdots (2m-1)} = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2m-2)^2 \sqrt{2m}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (2m-1)} \frac{(2m)^2}{2m \cdot 2m}$$

So

$$\sqrt{\pi/2} \approx \frac{2^{2m}(m!)^2}{(2m)!} \cdot \frac{1}{2m}$$

and inverting, we have

$$\frac{(2m)!}{2^{2m}(m!)^2} \approx \frac{1}{\sqrt{\pi m}}$$

Historical Note: Wallis discovered this infinite product for $\pi/2$ before calculus was invented. He did not know how to integrate by parts and had no clue about integrating trig functions. He guessed the answer by looking at $\int_0^1 (1-x^2)^m \, dx$ for integer m, finding a recursion relation that worked, and using it also for half-integral m.