MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Outline #8 (Two Continuous Random Variables)

Last modified: November 29, 2004
References:

- PRP, Chapter 4, sections 4.6 through 4.8.
- EP, sections 8.1 through 8.6.
- one handwritten page of notes (attached) reviewing the use of Jacobians in changing variables in a double integral.

1. As an illustration of the difficulties that arise in defining conditional expectation in the continuous case, consider the following example:

   As part of a land-reform program, small portions of a large circular region of land are being distributed to peasants who voted correctly in a recent election. Near the center of the region, the land is very fertile, but the fertility drops to almost zero at the perimeter.

   Peasants A and B are told that they are to receive parcels of land whose centers are at randomly-chosen locations on a path that satisfies the equation $y = x$.

   Peasant A says, "Fine, choose the center so that the ratio $\frac{y}{x}$ is between 1 and 1.01.

   Peasant B says, "Fine, choose the center so that the difference $y - x$ is between 0 and 0.01.

   Show that peasant B is more likely to receive a fertile plot.

2. Explain the limiting process that is needed to define the conditional distribution function and the conditional density function for random variable Y given $X = x$. As an example, let X and Y have the joint density function

$$f(x, y) = \frac{1}{x}, 0 \leq y \leq x \leq 1$$

   . and calculate $f_X(x)$ and $f_{Y|X}(y, x)$.

3. Define the conditional expectation of random variable Y given $X = x$. Prove that $\mathbb{E}(\mathbb{E}(Y|X)g(X)) = \mathbb{E}(Yg(X))$ and apply this formula to the special case where $g(x) = 1$ and
$f_{X,Y}(x, y) = \frac{1}{x}, 0 \leq y \leq x \leq 1$, in two ways:

   - Calculate $\mathbb{E}(Y)$ as a double integral, using the law of the unconscious statistician. Use polar coordinates.
   - Calculate $\mathbb{E}(Y|X = x)$.
   - Calculate $\mathbb{E}(Y)$ by conditioning on X, as

$$\int_0^1 \mathbb{E}(Y|X = x) f_X(x) dx$$

.

4. Continuing the same example, let $Z = X^2 + Y^2$.

   - Calculate one value of the distribution function for Z, $F_Z(1) = \mathbb{P}(X^2 + Y^2 \leq 1)$.
   - Calculate one value of the conditional distribution function for Z, $F_Z(1) = \mathbb{P}(X^2 + Y^2 \leq 1|X = x)$, and verify that

$$\mathbb{P}(Z \leq 1) = \int_0^1 \mathbb{E}(Z|X = x) f_X(x) dx$$

.

5. Suppose that random variables X_1 and X_2 are functions of random variables Y_1 and Y_2.

$$X_1 = x_1(Y_1, Y_2)$$
$$X_2 = x_2(Y_1, Y_2)$$

If the density function for X_1 and X_2 is $f_{X_1,X_2}(x_1, x_2)$, then the density function for Y_1 and Y_2 is

$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1(y_1, y_2), x_2(y_1, y_2))|J(y_1, y_2)|$ where

$J(y_1, y_2)$ is the Jacobian determinant

$J(y_1, y_2) = \frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}$.

Apply this rule to re-express the density function

$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$

as a density function for polar coordinates

$f_{R,\Theta}(r, \theta)$

6. Suppose that X and Y are uniformly distributed in the square $0 \le x, y, \le 1$.

   Show that X and Y are independent.

   Find the density function for $X + Y$ by setting

   $U = X + Y, V = X - Y$

7. Suppose that X and Y are uniformly distributed in the triangle

   $0 \le x, y, \le 1, x + y \le 1$.

   Show that X and Y are not independent.

   - Find the density function for $X + Y$ by setting $U = X + Y, V = X - Y$ and integrating over v. Explain why U and V are not independent. Calculate the conditional density $f_{U,V}(u, v)$.
   - Find the density function for $X + Y$ by setting $U = X + Y, V = \frac{Y}{X}$ Explain why U and V are independent.

8. Suppose that X and Y are uniformly distributed in the quarter-circle $0 \le x, y, \le 1, x^2 + y^2 \le 1$. Show that X and Y are not independent.

   Set

   $R = \sqrt{X^2 + Y^2}, \Theta = \arctan(\frac{Y}{X}$ and integrating over θ. Show that R and Θ are independent. Calculate the density function $f_{R,\Theta}(r, \theta)$ and the distribution function $f_{R,\Theta}(r, \theta)$

9. Analyze cases where X and Y are independent exponential variables, both with parameter λ. (example 7-11 in section 4.7 of PRP).

   - Use the change of variables $U = X + Y, V = \frac{Y}{X}$. Show that U and V are independent, and identify the density function for each. Integrate over v to show that U has the $\Gamma(2)$ distribution.
   - Use the change of variables $U = X + Y, V = X = Y$. Show that U and V are not independent. Again integrate over v to show that U has the $\Gamma(2)$ distribution.

10. By computing $\mathbb{P}(X+Y \leq z)$, using variables $Z = X+Y$, $U = X$, show that the density function for the sum of random variables X and Y is $\int_{-\infty}^{\infty} f_{X,Y}(u, z-u)du$, and that if X and Y are independent this becomes the convolution $\int_{-\infty}^{\infty} f_X(u)f_Y(z-u)du$. Consider the simple case where X and Y are uniformly distributed on [0,1], and explain the similarity to the discrete distribution that results from throwing two dice.

11. By computing $\mathbb{P}(X+Y \leq z)$, using variables $Z = X+Y$, $V = X-Y$, show that the density function for the sum of random variables X and Y can also be expressed as $\frac{1}{2}\int_{-\infty}^{\infty} f_{X,Y}(\frac{z-v}{2}, \frac{z+v}{2})dv$.

12. Use the convolution formula to show that if independent random variables X and Y have the exponential distribution with equal λ, then $X+Y$ has the $\Gamma(2)$ distribution.

13. Show that if both X and Y have the normal distribution (even with unequal variances) so does $X+Y$.