

MATHEMATICS 191, FALL 2004  
MATHEMATICAL PROBABILITY  
Outline #9 (Geometrical Probability)

Last modified: December 9, 2004

References:

- PRP, Section 4.13
- EP, sections 7.9, 7.13, 7.15, 8.12, 8.13

All of this topic is relevant material for the quiz on Thursday, December 16. Be sure that you master Crofton's method!

1. Bertrand's paradox: Describe three different reasonable interpretations of "choose a random chord of a circle of unit radius," and show that each interpretation leads to a different value for the probability that an equilateral triangle that has the random chord as one side lies within the circle. Here are the three interpretations:
  - The midpoint of the chord is a random variable with uniform density inside the circle. Then  $\mathbb{P}(\text{triangle fits}) = \frac{3}{4}$ .
  - Choose a point at random with uniform density on the circumference of the circle. Then form the radius of the circle joining the center to this point, choose a point at random with uniform density on this radius, and use it as the midpoint of the chord. This is more likely to lead to a long chord that passes close to the center of the circle. Now  $\mathbb{P}(\text{triangle fits}) = \frac{1}{2}$ .
  - Choose two points  $A$  and  $B$  independently with uniform density on the circumference of the circle, and let  $AB$  be the "random chord." Now  $\mathbb{P}(\text{triangle fits}) = \frac{2}{3}$ .
2. The midpoint of the chord in Bertrand's paradox is described by a pair of random variables  $R$  and  $\Theta$  that specify its polar coordinates. Specify the density function  $f_{R,\Theta}(p, \theta)$  for each of the three cases above.

3. Buffon's needle:

- A needle with length  $L$  is placed on the  $x$  axis, with its midpoint at the origin.
- In a circle of diameter  $d$ , centered at the origin, a point is chosen at random in accordance with the second interpretation of Bertrand's paradox.
- A grid of lines separated by distance  $d$  is laid down so that the stripes are perpendicular to the radius that was chosen and the one line that cuts the circle passes through the "random point" that was chosen.
- Show the the probability that the needle intersects the grid line that cuts the circle is  $\frac{2L}{\pi d}$ .
- With  $L = d$ , this is a famous and very inefficient way to determine the numerical value of  $\pi$ . Traditionally, the grid of lines (a flag) is held fixed and the needle is tossed "at random" onto the flag.

4. Here is a simple illustration of a trick for finding variances without doing any integration. Clever physicists sometimes use this technique to calculate the moment of inertia of a rigid rod about its center.

- Random variable  $X$  has values  $\frac{L}{4}$  and  $-\frac{L}{4}$ , each with probability  $\frac{1}{2}$
- Random variable  $Y$  is uniform in  $[-\frac{L}{4}, \frac{L}{4}]$
- Random variable  $Z = X + Y$  is then uniform in  $[-\frac{L}{2}, \frac{L}{2}]$
- $X$  and  $Y$  are independent; so  $\text{var}(Z) = \text{var}(X) + \text{var}(Y)$ . But clearly  $\text{var}(Z) = \alpha L^2$  and  $\text{var}(X) = \alpha (\frac{L}{2})^2$ . Determine  $\alpha$  algebraically.

5. As another example of the same sort of approach, suppose that  $X$  and  $Y$  are chosen uniformly and independently in  $[0, z]$ . Set  $\mathbb{E}(|Y - X|) = \alpha z$ . Now condition on whether  $X$  and  $Y$  are in the same half of the interval (in which case we have a scaled version of the same problem) or in different halves (in which case, if  $Y$  is in the right half,  $\mathbb{E}(|Y - X|) = \mathbb{E}(Y - X) = \mathbb{E}(Y) - \mathbb{E}(X) = \frac{3}{4}z - \frac{1}{4}z = \frac{1}{2}z$ .)

6. A different approach to the same problem, which also involves conditional probability and scaling, is Crofton's method. Let  $\mathbb{E}(|Y - X|) = S(z)$ , where the function  $S(z)$  is to be determined by solving a differential equation.

- Think about  $S(z + h)$ , where  $h$  is small and we will let it approach zero. There are two cases: either one of the points is in  $[z, z + h]$  (which occurs, for small  $h$ , with probability  $2\frac{h}{z}$ ) or both points are in  $[0, z]$ .
- By letting  $h \rightarrow 0$ , show that  $S(z)$  satisfies the linear differential equation

$$\frac{dS}{dz} + 2\frac{S}{z} = 1$$

- The classic way to such a linear differential equation is to search for an "integrating factor"  $\mu(z)$  for which the equation reduces to

$$\frac{d(\mu S)}{dz} = \mu$$

which is trivial to solve by integrating over  $z$ . By writing

$$\mu \frac{dS}{dz} = \frac{d(\mu S)}{dz} - S \frac{d\mu}{dz},$$

show that  $\mu(z) = z^2$  does the job in this case, and that  $S(z) = \frac{z}{3}$ .

7. As a more challenging application of the techniques just introduced, we solve problem (6) on page 136 of PRP. Ultimately we will show that if points  $P$ ,  $Q$ , and  $R$  are chosen independently and uniformly inside triangle  $ABC$ , then  $\mathbb{E}|PQR| = \frac{1}{12}\mathbb{E}|ABC|$ , where  $|PQR|$  denotes the area of triangle  $PQR$ .

As a first step, show that if  $\vec{W}$  is a random variable, the vector from vertex  $A$  to a randomly chosen point in  $ABC$ , then  $\mathbb{E}(\vec{W}) = \vec{G}$ , the vector from  $A$  to the center of gravity of triangle, the point where the medians all intersect. An easy way to do this is to write

$$\vec{W} = S\vec{u} + T\vec{v}$$

where  $\vec{u}$  is from  $A$  to the midpoint  $M$  of side  $BC$  and  $\vec{v}$  is from  $M$  to  $C$ . Then the density function  $f_{S,T}(s, t) = 1$  for  $0 \leq s \leq 1$ ,  $-s \leq t \leq s$ .

8. The next step is to show that if  $P$  and  $Q$  are chosen at random inside the triangle,  $\mathbb{E}|APQ| = \frac{4}{27}\mathbb{E}|ABC|$ . The trick is to condition on whether  $P$  and  $Q$  are both in  $ABM$  or both in  $AMC$ , which leads to a scaled version of the same problem, or whether they are in separate halves of the triangle, in which case the expectation is the area of the triangle formed by  $A$  and the centers of gravity of  $ABM$  and  $AMC$ .

9. The next step is to show that if  $P$  and  $Q$  are chosen at random inside the triangle while  $R$  is chosen randomly on side  $BC$ , then  $\mathbb{E}|PQR| = \frac{1}{9}\mathbb{E}|ABC|$ . For fixed  $R$ , the trick is to condition on whether  $P$  and  $Q$  are both in  $ABR$  or both in  $ARC$ , which leads to the problem that was just solved, or whether they are in separate halves of the triangle, in which case the expectation is the area of the triangle formed by  $A$  and the centers of gravity of  $ABR$  and  $ARC$ . Then integrate over the location of  $R$ , which is uniform on side  $BC$ .
10. To finish the solution, use Crofton's method. Since the ratio of areas is invariant under affine transformations, it is sufficient to consider a right isosceles triangle with vertices at  $(0,0)$ ,  $(0,x)$ , and  $(x,0)$ . Call its area  $A(x)$ . Now think about  $A(x+h)$ , where  $h$  is small. The probability that any one of the points  $P$ ,  $Q$ , or  $R$  is in the triangle with side  $x$  is  $\frac{x^2}{(x+h)^2}$ . Cube this to get the probability that points  $P$ ,  $Q$ , or  $R$  are all in the triangle with side  $x$ , in which case the function  $A(x)$  solve the problem. Otherwise one of the points is on the hypotenuse, which leads to the problem that was just solved. Letting  $h \rightarrow 0$  leads to the differential equation

$$\frac{dA}{dx} + 6\frac{A}{x} = \frac{x}{3}$$

The integrating factor, found as before, is  $\mu(x) = x^6$ , and the solution is  $A(x) = \frac{1}{24}x^2 = \frac{1}{12}|ABC|$ . SO in general  $\mathbb{E}|PQR| = \frac{1}{12}|ABC|$ .

11. As a simple corollary of the result just proved, solve Sylvester's problem: show that if four points are chosen independently and uniformly inside a triangle, the probability that no one of them lies inside the triangle formed by the other three (i.e. that they form a convex quadrilateral) is  $\frac{2}{3}$ .