MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Outline #5 (Discrete Random Variables and Expectation)

Last modified: October 13, 2004
References:

- PRP, Sections 3.1 through 3.7
- EP, Chapter 4 and sections 5.1 through 5.5

1. Write down the probability mass functions for the binomial and Poisson distributions, and show that the associated distribution functions meet the requirements of a discrete distribution function. Show how to convert any convergent infinite series of positive terms into a mass function, for example, Euler's famous

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} = \frac{\pi^4}{90}.$$

2. Define what it means for discrete random variables to be independent. Now suppose that a coin is tossed N times, where the random variable N has the Poisson distribution. The number of heads X and the number of tails Y are also random variables. Show that they are independent.

3. Define the expectation of a random variable X. Show that in the carnival game of "Chuck-A-Luck." the probability of winning is less than $\frac{1}{2}$ but the expected number of occurrences of the chosen number is $\frac{1}{2}$.

   Details of 'Chuck-A-Luck."
   You pick a number from 1 to 6. Three fair dice are rolled. If any comes up with your number, you win. What is the probability of winning?

   The number of losing rolls is 5 x 5 x 5 = 125.
   The total number of rolls is 6 x 6 x 6 = 216
   So there are 216-125 = 91 winning rolls, and your chance of winning is 91/216.

   Sanity check using expectation: Let the random variable X be the number of occurrences of your number on the three dice.

   - 1 roll with 3 of your number, so $\mathbb{P}(X = 3) = \frac{1}{216}$
   - 15 rolls with 2 of your number (3 places for the other number x 5 choices for it), so $\mathbb{P}(X = 2) = \frac{15}{216}$
   - 75 rolls with 1 of your number (3 places for it, and 5 x 5 pairs of other numbers for the other two dice), so $\mathbb{P}(X = 1) = \frac{75}{216}$
     $\mathbb{E}(X) = 3 \times \frac{1}{216} + 2 \times \frac{15}{216} + 1 \times \frac{75}{216} = \frac{1}{2}$

4. State and prove the "law of the unconscious statistician." Give an example to show that using this law can lead to different arithmetic than strict application of the definition of expectation. Then use it to show that expectation is a linear function on the space of random variables:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

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Here is a simple example. You are the curator of a small museum and have instructed your staff to count visitors as they enter and as they leave. The random variable X is the excess of entrances over exits. You determine that $\mathbb{P}(X = -2) = .05$
$\mathbb{P}(X = -1) = .1$
$\mathbb{P}(X = 0) = .5$
$\mathbb{P}(X = 1) = .2$
$\mathbb{P}(X = 2) = .15$

The random variable of interest to you is $Y = X^2$.

The "correct" way to calculate the expectation of Y is $\mathbb{E}(Y) = \sum y\mathbb{P}(Y = y)$. Since $\mathbb{P}(Y = 4) = .05 + .15 = .2$ and $\mathbb{P}(Y = 1) = .1 + .2 = .3$,

$\mathbb{E}(Y) = 4 \times .2 + 1 \times .3 + 0 \times .5 = 1.1$

A more direct calculation, using the law of the unconscious statistician, is

$\mathbb{E}(X^2) = \sum x^2\mathbb{P}(X = x) = 4 \times .05 + 1 \times .1 + 0 \times .5 + 1 \times .2 + 4 \times .15 = 1.1$

5. Define variance, and prove that var(X)=$\mathbb{E}(X^2) - \mathbb{E}(X)^2$

6. Define what it means for random variables to be uncorrelated. Prove that independent random variables are uncorrelated, and give an example to show that the converse is not necessarily true. Prove that for uncorrelated random variables X and Y, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$, but show that variance is not a linear function on the space of random variables.

   A simple example of variables that are uncorrelated but not independent.

   You answer two true-false questions. Each counts +1 if correct, -1 if incorrect. Random variable X is your total score. Random variable Y is your improvement: the score on the second minus the score on the first.

   These variables are not independent (assuming the probability of being right is not 0 or 1) because, for example, if X=2 then $Y = 0$.

   However, the random variable XY is equal to 0 for every outcome, since if your score X is +2 or -2 your improvement Y is 0 and otherwise your score is 0. So its expectation is 0.

7. Give an example of a random variable whose expectation is undefined because the infinite series that defines it is only conditionally convergent, not absolutely convergent.

   Try this: X is the payoff in a game.

   $\mathbb{P}(1) = \frac{8}{\pi^2}$
   $\mathbb{P}(-3) = \frac{8}{9\pi^2}$
   $\mathbb{P}(5) = \frac{8}{25\pi^2}$
   $\mathbb{P}(-7) = \frac{8}{49\pi^2}$
   The probabilities sum to 1, as you can prove by starting with Euler's

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$$

   and subtracting $\frac{1}{4}$ of the series to get rid of the even-numbered terms.

   The expectation looks like $\frac{2}{\pi}$, but it is really undefined because the series that defines it is not absolutely convergent. The variance is infinite.

8. Give an example of a random variable whose expectation is infinite but for which a reasonable person would pay only a finite amount to play a game whose payoff is given by this random variable. This is the "St. Petersburg paradox" from p. 55 of PRP.

9. Using indicator functions, prove that if n letters are placed at random into n matching envelopes, the probability that no letter ends up in the correct envelope approaches e^{-1} as $n \to \infty$. Show that the same result follows directly from Waring's theorem.

10. Generalize this result to the case where precisely k of the n letters end up in the correct envelope.

11. Explain the application of the "probabilistic method" that is described on page 59 of PRP, and show that it is really just an example of counting something in two alternative ways. Give a simpler example (four letters in three mailboxes) that illustrates more clearly the connection with the pigeonhole principle.

12. Show how the Poisson distribution can be obtained as a limiting case of the binomial distribution, and derive the expectation and variance for the Poisson distribution by this limiting process.

13. Describe a process that leads to a random variable with a geometric distribution and a generalization of this process that leads to a negative binomial distribution. Calculate the expectation and variance for the negative binomial distribution by treating it as a sum of independent random variables.

14. Prove that two random variables are independent if and only if their joint mass function $f(x, y)$ can be expressed as a product $g(x)h(y)$. Show how this fact explains the independence of the Poisson random variables X and Y in item 2 above.

15. Define the correlation $\rho(X, Y)$ of two random variables. By proving the Cauchy-Schwarz inequality, show that $|\rho(X, Y)| \leq 1$, with equality only if there is a linear relationship between X and Y.

16. For two random variables X and Y, define the conditional distribution function, conditional mass function, and conditional expectation of Y given $X = x$. Prove that $\mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(Y)$

17. (This is the same as the example on p. 68 of PRP, with a different story line)
High-school seniors apply to N colleges, where N has the Poisson distribution with parameter λ. Each applicant is admitted by each college independently with probability p. Calculate the following:

    (a) the mean number of colleges to which students applied.

    (b) the mean number of acceptances received by students who applied to N colleges.

    (c) the mean number of acceptances per student.

    (d) the mean number of applications filed by students who were accepted at K colleges.