MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Outline #1 (Countability and Uncountability)

Last modified: September 16, 2004
Reference: Apostol, Calculus, Vol. 2, section 13.19 (attached).

The aim of these lecture outlines is to be complete enough so that a group of students, armed with the outline and the textbooks, could give the lectures. The more thorough the treatment in the textbooks, the shorter the notes in the outline will be.

Outlines should be useful for review, but they will omit lots of details that will be presented in lecture.

1. Functions from one set to another.

   Explain the two requirements for two sets to be in one-to-one correspondence. Illustrate these requirements (and how they can fail) for the case of finite sets with just three or four elements. Then show that the infinite set of positive integers is in one-to-one correspondence with the set of even positive integers by exhibiting a bijection from the first to the second, and describe the practical application of creating space in "Hilbert's hotel."

   Notes:

   Here is some useful terminology that relates to the definition on p. 501 of Apostol. A function that satisfies (a) is called "onto" or "surjective." A function that satisfies (b) is called "one-to-one" or "injective." A function that satisfies both (a) and (b) is called "bijective" or "invertible." Two sets that are in one-to-one correspondence are called "equivalent."

   Sets A and B are in 1-to-1 correspondence if there exists a function $f : A \longrightarrow B$ such that

   - The range of f is all of B (f is surjective)
   - If x and y are distinct elements of $A, f(x)$ and $f(y)$ are distinct elements of B (f is injective).

     Thus f is both surjective and injective, hence bijective.

   Such a function f has an inverse g, and the sets A and B are called equivalent: $A \sim B$ and $B \sim A$

   Example: the set B of even positive integers is equivalent to the set A of all positive integers. The bijection $f : A \longrightarrow B$ is $f(n) = 2n$

   A set that is equivalent to $\{1, 2, ...n\}$ is a finite set with n elements.

The legendary "Hilbert's Hotel" has an infinite number of rooms, all filled. The manager learns that because of an evacuation caused by a hurricane, a large number of additional guests are about to arrive. She quickly moves the guest in room 1 to room 2, the guest formerly in room 2 to room 4, etc., thereby freeing up all the odd-numbered rooms for the new arrivals!

2. Countably infinite sets.

   Define the terms "countably infinite set" and "countable set." Then use this definition to show that the set of all integers (positive, negative, or zero) is countably infinite. Also show directly from the definition (without using other properties as Apostol does in Example 3 on p. 503) that the collection of all two-element subsets of the positive integers is countably infinite, as is the collection of all three-element subsets of the positive integers. In a similar way, show that the set of positive rational numbers is countably infinite.

   Notes: A set that is equivalent to {1,2,3...} is a countably infinite set.
   A computer version of this concept: Imagine a function that produces an infinite sequence of values, all in set B. If it produces each element of B once and only once, then B is countably infinite. If, given any supposed such program, you can invent an element of B that the function can never produce, then B is uncountable.

   Doing the examples:

   - A bijection from the integers to the positive integers,

     $$f : \mathbb{Z} \longrightarrow P,$$

     is $f(n) = 2n$ for positive n, $f(n) = 1 - 2n$ otherwise.
   - The family of 2-element subsets of the positive integers is countably infinite.

     Proof:

     Alternative 1: Arrange the sets in order of increasing sum of elements, so the computer program produces the bijection
     1: $\{1, 2\}$
     2: $\{1, 3\}$
     3: $\{2, 3\}$
     4: $\{1, 4\}$
     5: $\{1, 5\}$
     6: $\{2, 4\}$
     7: $\{1, 6\}$
     8: $\{2, 5\}$
     9: $\{3, 4\}$
     and so on. Clearly any 2-element set will appear once and only once.

Alternative 2: Arrange the sets in order of increasing larger element, so the computer program produces the bijection
1: $\{1, 2\}$
2: $\{1, 3\}$
3: $\{2, 3\}$
4: $\{1, 4\}$
5: $\{2, 4\}$
and so on. Clearly any 2-element set will appear once and only once

- The set of positive rational numbers is countably infinite:
  Proof: Use the "sum" approach (alternative 1 above) to generate all ordered pairs {numerator, denominator}. Omit any that lead to fractions that are not in lowest terms.
  1: 1/1
  2: 1/2
  3: 2/1
  4: 1/3
  5: 3/1 (note – 2/2 was omitted)
  6: 1/4
  7: 2/3
  8: 3/2
  9: 4/1
  10: 1/5
  11: 5/1 (note: 2/4, 3/3, and 4/2 were all omitted)
  and so on.

  To prove "countably infinite." you still have to rule out the possibility that the rationals are a finite set. (For example, if any fraction where the sum of numerator and denominator exceeded 1 million were not in lowest terms, there would only be finitely many rational numbers.) Fortunately, the rational numbers contain an infinite subset, the positive integers, so they cannot be a finite set.

The Windows application enum.exe (available on the course Web site and in class) makes the definition of "countably infinite" vivid. For two-element subsets, everyone should choose and write down such a set before the program starts. A chosen set will show up once and only once as the program runs, and it can be predicted exactly where on the list it would appear.

Properties of countable sets: proofs were left as exercises by Apostol but are included below:

a Every subset of a countable set is countable

b The intersection of ANY collection of countable sets is countable

c The union of a COUNTABLE collection of countable sets is countable d The Cartesian product of a FINITE number of countable sets is countable

3. Subsets and intersections.

   Present the solutions to exercises 3 and 4 on p. 505 of Apostol, thereby establishing properties (a) and (b) on p. 502. As an example, show that the set of prime numbers is countable. You do not need to show that it is countably infinite, just countable!

   Notes: In dealing with the finite case, prove that every subset of a finite set is finite. Then you can state the obvious but useful corollary(already invoked in the case of the rational numbers) that if a set has a countably infinite subset, it is infinite.

   If Apostol's hint for exercise 4 is not quite enough, just remember that the intersection of a set A with any other set B must be a subset of A.

4. Cartesian products

   Present the solution to exercise 5 on p. 505 of Apostol, thereby establishing property (d) on p. 502.

   Notes:

   The inductive proof that Apostol suggests is a nice way to proceed, but you can also just extend his prime-number trick. For example, to show that the set of all sequences of 5 positive integers like (m, n, r, s, t) is countable, let $f(m, n, r, s, t) = 2^m 3^n 5^r 7^s 11^t$.

5. Countable unions.

   Present the solution to exercise 6 on p. 505. This shows most of property (c) on p. 502, namely that the union of a countable collection of disjoint countable sets is countable. The extension to sets that are not disjoint is unenlightening.

   Finally, present examples 3 and 4 on p. 503.

   Notes:

   Here is an alternative direct proof for 3, "The family of n-element subsets of the positive integers is countable." Enumerate all n-element sets whose sum is 1, then 2, then 3, then 4, etc. This is a countable union of finite sets, hence countable.

   The proof for 4, "The collection of all FINITE subsets of a countable set is countable", uses property (c). This is a countable union of countable sets, one for each value of n.

6. Uncountability of the real numbers.

   Present example 6 on p. 505 of Apostol. This is the simplest and best-known form of Cantor's ingenious argument.

Notes: Here is a nice way of looking at Cantor's argument. Suppose that someone claims to have written a computer program that enumerates all the real numbers between 0 and 1 in a list, which might begin like this:

1. $\pi - 3$
2. $3/7$
3. $\sqrt{2} - 1$
4. ....

In terms of decimals the list will look like this:

1. .141592....
2. .428571428...
3. .414213562....

You construct a number as proposed by Apostol. For the given example it will begin .211.... Since it differs in at least one decimal place from each number on the list, it cannot appear anywhere on the list; so the author of the program is making an incorrect claim.

This is how all proofs of uncountability work. Assume that the set is countable, and show that this assumption leads to a contradiction.

7. Uncountability in general.

Present Example 5 on pp. 503-504 of Apostol. The key result is that a function from a set A to the collection of all subsets of that set cannot be onto (surjective).

Notes: The way you prove this is to assume the existence of a surjective function f and to show that this assumption leads to a contradiction. Before presenting Apostol's proof it is worth doing the example of a finite set, say with four elements. In this case the result is obvious, since there are 16 different subsets of {1,2,3,4}, but suppose you ask the class to choose $f(1), f(2), f(3)$, and $f(4)$.

Suppose you end up with $f(1) = \{1, 2\}$
$f(2) = \{3\}$
$f(3) = \{1,2,4\}$
$f(4) = \{1,2,3,4\}$

Now you note that $f(1)$ includes 1 and $f(4)$ includes 4, but $f(2)$ fails to include 2 and $f(3)$ fails to include 3. So you construct the set $B = \{2, 3\}$, which cannot be in the range of f. The fact that you can always do this shows that no set of choices can create a surjective f. This is Apostol's argument, and it works for a countably infinite set as well as for a finite set.

Here is a summary of Apostol's argument.

The collection of ALL subsets of a countably infinite set is UNcountable

Proof (by contradiction): Imagine a computer program that generates a sequence of all subsets of the positive integers: Sometimes the nth element in this sequence will include the integer n; sometime it will not. Let B be the set of all integers n for which the program generates a subset that does not include n.
Assume that the function generates B as the bth element in the sequence. Either b is an element of B, or it is not.
Now show that either alternative implies its opposite!

First, assume that b is an element of B.
But the definition of B is that b is not an element of B, so if b is an element of B then b is not an element of B

Second, assume that b is not an element of B. But then b satisfies the criterion for membership in B, so if b is not an element of B then b is an element of B. So either assumption about b leads to a contradiction. Thus no such b can exist, and B is uncountable.

8. State Borel's normal number theorem (but do not prove it).

   Notes:

   A number is called "normal" if, when it is expanded in any number system, all the digits occur with equal frequency. If π is normal, all the digits 0-9 should show up equally often. Experimental results for 1.24 trillion digits are consistent with this hypothesis. The following are equivalent:

   - Flip a fair coin infinitely many times.
   - Choose a real number between 0 and 1 at random.

   For the coin flips we will be able to show that a significant excess of heads over tails (or vice versa) is highly improbable. So non-normal random real numbers should be highly unusual, even though there are uncountably many of them.