MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Outline #7 (Continuous Random Variables)

Last modified: November 8, 2004

References:

- PRP, Chapter 4, up through section 4.5.
- EP, sections 7.2 through 7.4.

Only topics 1 through 3 are relevant to the quiz on Thursday, Nov. 18.

1. State and prove the properties of a density function $f_X(x)$, and exhibit the density function for the arc sine distribution.

2. Define the expectation of a continuous random variable. By interchanging order of integration show that, for a random variable X that cannot be negative (the density function $f(x) = 0$ for $x < 0$), the expectation can be calculated by the formula

$$\mathbb{E}(X) = \int_0^\infty (1 - F(y))dy = \int_0^\infty \mathbb{P}(X > y)dy$$

3. State the "law of the unconscious statistician" for the continuous case, and prove it in the special case of a function $g(X)$ which takes only non-negative values. Illustrate this by calculating the expectation of x and $\sin x$ for a random variable X that is uniformly distributed in $[0, \frac{\pi}{2}]$.

4. Write down a distribution function (if possible) or a density function for the following important cases:
   - the uniform distribution.
   - the exponential distribution (reminiscent of the discrete geometric distribution)
   - the normal distribution (reminiscent of the discrete binomial distribution).
   - the gamma distribution (reminiscent of the discrete Poisson distribution).
   - the Cauchy distribution.

5. Prove that a random variable Y with a normal distribution whose density function is

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

is properly normalized and has a mean of zero and a variance of 1.

6. Prove that a random variable X with a normal distribution whose density function is

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is properly normalized and has a mean of μ and a variance of σ.

7. Explain how to calculate the distribution and density functions for a random variable $Y = g(X)$ from the density function for X, and give examples of how this can be used to generate other continuous distributions from uniform and normal distributions.

8. Show how, if a distribution function $F(x, y)$ is differentiable enough times, you can calculate the density function by taking partial derivatives. Then prove that continuous random variables X and Y are independent if and only if their joint density function is of the form $f(x, y) = g(x)h(y)$.

9. State the "law of the unconscious statistician" for the case of two random variables, and use in to show that the expectation of a linear combination of two random variables is the same linear combination of their expectations.

10. Write the density function for the bivariate normal distribution. Show that the quantity ρ in equation 10 on p. 100 of PRP is both the correlation and the covariance of the random variables X and Y.