MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Assignment #9

Problems to be discussed in section on Monday, Dec. 6: All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1,3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 4.4, problem 6. (expectation for a normal distribution)
2. Section 4.5 , problem 4. (expectation and covariance for min and max of two independent random variables)
3. Section 4.6, problem 7. (conditional expectation for correlated variables)
4. Section 4.8, problem 3. (density function for a sum) Be careful with the limits of integration.
5. Section 4.14, problem 35. (secretary/marriage problem)
6. Section 4.7, problem 1. (three independent, uniform random variables) The "arguing more directly" solution is very easy if you reduce it to calculating the volume of a solid whose top bounding surface is the graph of $z = \sqrt{xy}$. This gives the probability for the complement of the desired event.

Problems to be handed in on Thursday, December 9:

This set is short because there were problems due on Tuesday.

1. You possess one nucleus each of two different species of radionuclide. The first has decay constant λ, which means that the probability that it will survive for time t is $e^{-\lambda t}$. (Note: this is 1 minus the distribution function, but the density function is also exponential.) The second has decay constant 2λ. Let random variable T_1 be the time after which the nucleus of species 1 decays; let random variable T_2 be the time after which the nucleus of species 2 decays.

   The carelessly stated question is "if both nuclei decay at the same time, when is the decay likely to occur?"

   (a) Find a density function, the expectation, and the variance for $U = T_1 - T_2$. Use $V = T_1$ as your second variable.

   (b) Find a density function and the expectation for $V = T_1$, conditioned on $U = T_1 - T_2 = 0$.

   (c) Find a density function for $Z = \frac{T_1}{T_2}$. Use $W = T_1$ as your second variable. Find the expectation of Z(which may be infinite.)

   (d) Find a density function and the expectation for $W = T_1$, conditioned on $Z = \frac{T_1}{T_2} = 1$.

2. (a) By using the change of variables

   $$Z = X + Y, V = X - \alpha Y$$

   for arbitrary $\alpha \geq 0$, derive a convolution formula for the density function $f_Z(z)$ that includes as special cases both the formula in the textbook and the alternative one derived in lecture.

   (b) By choosing $\alpha = \sigma^2$, show that if X is $N[0, \sigma^2]$ and Y is $N[0, 1]$, than $Z = X + Y$ is $N[0, \sigma^2 + 1]$.

3. Choose a "random point" within the unit circle by first choosing a random chord, then choosing a point at random (uniform density) on the chord. Find the expectation of the random variable $X^2 + Y^2$, the square of the distance of the point from the center of the circle. Solve the problem for each of the three ways of choosing a "random chord" in Bertrand's paradox (PRP, pages 133-134).

The following problem is modeled on an old quiz problem.

4. A U.S. commander in Iraq is given funds to assist with security and redevelopment of towns in his region. He decides that each town should receive a random aid allocation, with the random variable X being the number of megadollars for security and the random variable Y being the number of megadollars for reconstruction. The total of number of megadollars, $X + Y$, must satisfy $3 < X + Y < 6$, and the ratio of redevelopment aid to security aid, $\frac{Y}{X}$, can range from $\frac{1}{2}$ to 2. For values of X and Y that satisfy all these restrictions, the density function $f_{X,Y}(x, y) = c$, a constant. Otherwise it is zero.

   (a) Draw a diagram of the region in which $f_{X,Y}(x, y)$ is nonzero, and determine the value c.

   (b) What is the conditional density $f_{Y|X}(y, X = 2)$? If you can read this off the diagram, that is fine.

   (c) For what value of x is the conditional expectation $\mathbb{E}(Y|X)(X = x)$ a maximum? Again, if you can read this off the diagram, that is fine.

   (d) Are X and Y independent? Justify your answer.

   (e) Introduce new random variables that specify total aid and ratio of redevelopment funds to security funds, as follows:

   $$U = X + Y, \quad V = \frac{Y}{X}$$

   Calculate the density function $f_{U,V}(u, v)$, being careful to specify for what range of values of u and v it is nonzero. You may wish to check that the integral over this region is 1.

   (f) Are U and V independent? Justify your answer.

   (g) Calculate $\mathbb{E}(U)$ and $\mathbb{E}(V)$.