MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Assignment #8 (revised!)

Problems to be discussed in section on Monday, Nov. 29: Note: These were postponed from the week of Thanksgiving. All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1,3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 4.14, problem 2 (normalizing a density function).
2. Section 4.1 , problem 1b (normalizing another density function).
3. Section 4.3 , problem 3 (expectation of X^r).
4. Section 4.4, problem 3 (moments of the Cauchy distribution).
5. Section 4.3, problem 4 (relation among mean, median, and variance).
6. Section 4.4, problem 1 (gamma function for half-integers).

Problems to be handed in on Tuesday, December 7:

The last two problems will be much easier to do after you have seen Jacobians in action on December 2.

1. Suppose that X and Y are independent random variables. Each has a normal distribution with mean 0 and standard deviation σ. Let $Z = X^2 + Y^2$. This is a special case of the "chi-squared distribution."

   (a) By writing down an appropriate double integral and changing to polar coordinates, evaluate $\mathbb{P}(Z \leq z)$.

   (b) Determine the distribution function and density function for Z. You will have seen them before; identify the distribution. Sketch a graph of each.

   (c) Calculate by integration the mean and variance of Z. If you are feeling lazy, you can use Mathematica or tables to evaluate the integrals.

   (d) Let $R = \sqrt{Z}$. Determine the distribution function, density function, mean, and variance for R.

   (e) Suppose that a darts player throws darts in such a way that you the coordinates of the dart are given by X and Y as specified above. You want to make a target with two circles on it, so that the dart will hit in the inner circle $\frac{1}{3}$ of the time, between the circles $\frac{1}{3}$ of the time, and outside the outer circle $\frac{1}{3}$ of the time. Find approximate numerical values, in terms of σ, for the radius of the inner circle and of the outer circle. (You may need to use a calculator or Mathematica.)

2. Let X, Y be random variables uniformly distributed on (0,1). Let S be defined by $S = [X/Y]$ where [..] denotes the nearest integer function. So, for example, $[\frac{0.2}{0.5}] = 0$, $[\frac{0.8}{0.3}] = 3$.

   (a) By a geometric argument (or otherwise), calculate the probability $\mathbb{P}(a < X/Y < b)$ for a, $b > 1$.

   (b) Calculate the probability that $S = k$ for integers k=0,1,2,3.

   (c) What is the probability that S is even? You can sum the series if you know that 1 - $\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$

3. A darts player always throws his dart so that it hits within the circle of radius 1 centered on the bull's eye. Within this circle the probability density is given by $f_{X,Y}(x, y) = C(1 - x^2 - y^2)$.

   (a) What is the value of C?

   (b) Let $R = \sqrt{X^2 + Y^2}$. Calculate by integration in polar coordinates the distribution function and a density function for R.

   (c) Calculate the marginal distribution function $F_X(x)$ and a marginal density function $f_X(x)$. Remember, when you integrate over y, that the limits of integration depend on x.

   (d) What is the conditional density function $f_{Y|X}(y|X = x)$?

   (e) What is the conditional density function $f_{R|X}(r|X = x)$?

4. Suppose that random variables X and Y have a bivariate normal distribution as specified by equation 10 on p. 100 of PRP. Let $U = X - Y$ and $V = X + Y$. Find a joint density function for U and V, and calculate the variance of each and the covariance of U and V.

5. Suppose that X and Y are independent random variables whose density functions are both given by special cases of the gamma distribution, namely

$$f_X(x) = \lambda^2 x e^{-\lambda x}$$

$$f_Y(y) = \frac{\lambda^3}{2} y^2 e^{-\lambda y}$$

   . Find the joint density function $f_{U,V}(u, v)$ for the random variables

$$U = X + Y, V = \frac{X}{X + Y}$$

   Show that U and V are independent, and identify the marginal density functions $f_U(u)$ and $f_V(v)$