MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Assignment #6

Problems to be discussed in section on November 1:

All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1, 3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 3.11, problem 7. (lack of memory in geometric distribution)
2. Section 3.11, problem 10. (hypergeometric distribution)
3. Section 3.11, problem 11. (hypergeometric distribution from binomial)
4. Section 3.11, problem 13b (red and blue balls in an urn, second method only. If the "little effort" mentioned is too much for you, at least plug in the answer to show that it works.).
5. Section 3.6, problem 4 (two discrete random variables)
6. Section 3.9, problem 4 (problem of the points by recurrence).

Problems to be handed in on Thursday, November 4:

1. (This is an easy but instructive problem, based on the last section problem and problem 3.11.24. When we consider characteristic functions this "problem of the points" will arise yet again.)

   Pedro Martinez is sent to the plate with instructions to take pitches until he walks or is called out on strikes. Each pitch is independent, and the probability of a ball is p for each pitch. Calculate, as a function of p, the probability that Pedro will walk: that he will receive 4 balls before 3 strikes. Although there is no closed-form solution to this problem, you can get the answer as a sum of three terms in two different ways, and both should lead to the same polynomial in p.

   (a) Add together the probability for Pedro to walk on 4, 5, or 6 pitches, taking 0, 1, or 2 strikes before the final ball.

   (b) Assume that the pitcher throws 6 pitches, at which point Pedro has struck out if he has not walked. Add together the probability for 4, 5, or 6 of them to be balls.

   You may want to evaluate the answer for a few values of p, but you are not required to. If you do, consider using Mathematica.

2. Jeff Suppan is on third base and has taken a two-step lead towards home plate. Just 3 steps away, 5 steps from third, is the coach who will send him home with a run. A ground ball is hit to the right side of the infield, and Suppan begins a random walk. Each step he takes is toward the coach with probability $p = 0.6$ or back towards third with probability $q = 0.4$.

   Let p_k denote the probability (for $1 \leq k \leq 4$) that Suppan returns to third base before reaching the coach. Set up a recurrence relation for p_k as in example 6 on page 74 of PRP, and solve it with the boundary conditions $p_0 = 1$, $p_N = 0$ to calculate the probability p_2 that Suppan will return to third base before reaching the coach.

3. A convenience store offers Lottery scratch tickets for \$1, \$2, or \$4. When you scratch the ticket, with probability $p = \frac{1}{2}$ you receive double the cost of the ticket, otherwise you receive nothing. So buying a sequence of scratch tickets is a random walk.

   You enter the store with \$8, determined to keep buying tickets until you either achieve your target of \$12 or run out of money. Using the formulas derived in class and in the textbook, calculate the probability of achieving your target for the three cases of tickets that cost \$1, \$2, or \$4.

   Repeat the calculation for the more realistic assumption that $p = \frac{1}{3}$.

4. Harvard and Yale's football teams are so evenly matched the the probability of Harvard winning any game is exactly $p = \frac{1}{2}$. The teams agree to keep playing a series of games until one team has won three games more than the other.

   - What is the probability that each team will have been ahead in the series before it ends?
   - What is the probability that each team will have been ahead by two games in the series before it ends?

   The analysis of problem 3.11.32 in the text may be useful.

5. In class the ballot theorem was proved by a combinatorial approach using the reflection principle, but a recurrence approach is also possible. Suppose that in an election candidate George has m votes and candidate John has n votes, where $m > n$, and the ballots are arranged in a random order and recounted publicly. Let $p(m, n)$ denote the probability that George is ahead at every stage during the recount. There are two ways for this to happen: either the last ballot counted is for George, or it is for John. By conditioning on this last ballot (which is more likely to be for George), set up a recurrence for $p(m, n)$ and show that it is satisfied by $p(m, n) = \frac{m-n}{m+n}$ in accordance with the ballot theorem.

6. An impoverished student starts with a balance of \$0 in his bank account, which has a credit line so that the balance can go negative. Each day he either deposits \$1 with probability $\frac{1}{2}$ or withdraws \$1 with probability $\frac{1}{2}$. Show that the probability that after $2n$ days his balance is again \$0, without the credit line having been used, is

$$(\frac{1}{2})^{2n}\frac{(2n)!}{(n+1)(n!)^2}.$$

(Hint: after 1 day the balance must be +\$1. Any path that subsequently drops below \$0 matches a reflected path that has a balance of -\$3 after the first day.