MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Assignment #5

Problems to be discussed in section on October 25:
All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1,3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 3.3, problem 2. (Vice-Chancellors again)
2. Section 3.3, problem 3a. (teams rolling dice)
3. Section 3.4 , problem 9a. (matching from inclusion-exclusion)
4. Section 3.11, problem 6 (sum of independent Poisson variables)
5. Section 3.6, problem 6 (voter paradox)
6. Section 3.3 , problem 8. (stopping strategy)

Problems to be handed in on Thursday, October 28:

1. A Bambino attempts to place a curse on the Red Sox and succeeds with probability $\frac{1}{2}$. The Red Sox then play a seven-game series, with the outcome of each game being independent. If cursed, they win any game with probability $p_c = \frac{1}{3}$. If not cursed, being a superior team, they win any game with probability $p_{nc} = \frac{2}{3}$.
   (a) What is the probability that one team (either team) will win the first three games and that the other will win the last four games? Is this greater or less than the probability of the same event if the curse is not an issue and the probability of winning each game is $\frac{1}{2}$?
   (b) If the Red Sox have lost the first three games, what is the conditional probability that they were cursed? (Notice that you can only answer this question because you have a model for how the curse works.)
   (c) If the Red Sox have lost the first three games, what is the conditional probability that they will win the next four games? Is this greater or less than the probability of the same event if the curse is not an issue and the probability of winning each game is $\frac{1}{2}$?
   (d) If the Red Sox lose the first three games, then win the next four games, what is the conditional probability that they were cursed?

2. In general, random variables X and Y are said to be independent if $\{X \le x\}$ and $\{Y \le y\}$ are independent events for all pairs (x, y) (PRP, page 92). For discrete random variables, an equivalent statement is that $\{X = x\}$ and $\{Y = y\}$ are independent events for all pairs (x, y). Let

$$A = \{X < x, Y < y\}$$

$$B = \{X < x, Y \le y\}$$
$$C = \{X \le x, Y < y\}$$
$$D = \{X \le x, Y \le y\}$$
$$E = \{X = x, Y = y\}$$

   (a) Express D as a union of three of the other events. Draw a diagram with a lattice of dots, for the case where x and y assume only integer values, to illustrate your answer

   (b) Using inclusion-exclusion, express $\mathbb{P}(E)$ in terms of the probabilities of the other four named events.

   (c) Prove that if X and Y have the property that $\{X = x\}$ and $\{Y = y\}$ are independent events for all pairs (x, y), then $\{X \le x\}$ and $\{Y \le y\}$ are independent events for all pairs (x, y).

   (d) Prove the converse of part (c), perhaps using earlier results.

3. A necklace contains 23 pearls strung on a circular wire. 16 of them are worthless imitations, but the other 7 are worth \$1000 each. Use the "probabilistic method" or an equivalent counting argument to show that if you are allowed to choose a sequence of 10 consecutive pearls, you can always find one sequence that is worth at least \$4000. Can you also be sure of finding some sequence that is worth \$2000 or less?

4. A telemarketer makes a sequence of phone calls. After each phone call, he uses a chance device to decide whether to quit for the day (with probability a) or to make another call (with probability $1 - a$). So the number of calls N is a random variable with a geometric distribution. Each call is successful with probability p. The random variable K is the number of successful calls. Calculate the expected number of calls that were made, given that exactly k calls were successful. That is, calculate the conditional expectation $\mathbb{E}(N|K = k)$.

5. In each of two consecutive games, a baseball player has five at-bats. This player has probability $p = 0.3$ of getting a hit in any given at-bat. You learn that the player has made a total of 4 hits in the two games. Conditioned on this, what is the mass function for the number of hits X in the first game? Compute the expectation and variance of X.

6. You roll 6 fair dice and get a random number X of sixes. You then roll these 6 dice again and get a random number Y of sixes. The random variable $Z = X + Y$ is the total number of sixes from all 12 die rolls.
A is the event $X \geq 1$.
B is the event $Z \geq 2$.
Compute $\mathbb{P}(A), \mathbb{P}(B)$, and the conditional probabilities $\mathbb{P}(B|A)$,$\mathbb{P}(B^c|A)$,$\mathbb{P}(B|A^c)$, and $\mathbb{P}(B^c|A^c)$. Show that your answers agree with Isaac Newton's solution to problem 3.8.5 in PRP(attributed to Samuel Pepys).