MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Assignment #3

Because next Monday is a legal holiday, it would be a violation of Massachusetts law for sections to meet!

Problems to be handed in on Thursday, October 14:

The first two problems (both quite short) are taken from one of last year's quizzes.

1. (This problem is loosely based on some work with a recently-discovered 1774 census of Rhode Island)

   A genealogist, on analyzing names in 18th-century Rhode Island, has ascertained the following:

   The probability that a male child was a slave was 0.3.

   Male children who were slaves were given classical names like "Caesar" and "Aesop" with probability 0.6.

   Male children who were free were given classical names like "Caesar" and "Aesop" with probability 0.2.

   Event A is "the child was a slave" and event B was "the child had a classical name."

   (a) Using set-theoretic notation, express the event "the child was either free or had a classical name" in terms of events A and B and calculate its probability.

   (b) Are events A and B independent? Justify your answer.

   (c) The genealogist encounters the classical name "Cicero Greene". What is the conditional probability, given his name, that Cicero was a slave?

   (d) The genealogist learns that Cicero had one sibling, his brother Roger. Either both were slaves, or both were free. What is the conditional probability that both were slaves?

2. In the liberal state of Calitopia, monthly pre-tax personal income in thousands of dollars is a random variable X that is equally likely to have any value between 0 and 16. Income after tax is a random variable given by the function $Y = \sqrt{X}$, but Y is rounded up to 1 if it is too small and down to 3 if it is too large.

   (a) If $F_Y(x)$ is the distribution function for this random variable Y, what is $F_Y(1)$? What is $F_Y(2)$?

   (b) Sketch an accurate graph of $F_Y(x)$.

(c) Is Y a discrete random variable, a continuous random variable, or neither? Explain.

3. American cereal manufacturers, impressed by the upsurge in sales of Corn Flakes in the U.K., launch a new brand of Harvard Bran Flakes. Each box contains a bust of Nathan Pusey, Derek Bok, Neil Rudenstine, or Larry Summers, each with probability $\frac{1}{4}$. An eager high-school senior who has applied early-decision to Harvard, anxious to become the first kid on his block to get a complete set of busts, rushes to the supermarket and buys four boxes of Harvard Bran Flakes.

   Using the generalized inclusion-exclusion formula as in problem 1.3.4 of PRP, calculate the probability that the student gets in his collection of busts

   (a) at least one Summers

   (b) at least one Summers and at least one Rudenstine.

   (c) at least one Summers, at least one Rudenstine, and at least one Bok.

   (d) at least one of each of the four busts.

   Hint: look at the solution to 1.3.4 on p.137 of 1000Ex. The events to consider are A_1 = "no Summers," etc. Then take complements of unions of these. The last answer is easily checkable, since there are 4! ways of getting one of each bust.

4. Using the corollary of the inclusion-exclusion formula proved in lecture in conjunction with Waring's theorem (last line of p. 143 in 1000Ex), calculate the probability that the student gets

   - busts only of Summers, Rudenstine, and Bok.
   - busts only of Summers and Rudenstine.
   - busts only of Summers.

   Some of these answers are easily checkable, but please do the problem as specified.
   Finally, use Waring's theorem to calculate the probability N_k that the student gets precisely k different busts for k = 1,2,3,4. These probabilities should sum to 1, of course.

5. You have to deliver crucial supplies using airplanes with very unreliable engines. Each engine has a probability p of lasting for the entire flight, and engine failures are independent events. If half or more of the engines fail, the plane crashes. Your choice is between using two-engine planes, which crash if either engine fails, or 4-engine planes, which crash if two or more engines fail.

(a) What is the probability that exactly three engines on a four-engine plane will survive?

(b) Determine for what value of p the probability that a plane will not crash is the same for 2-engine and 4-engine planes.

(c) For this value of p, would 6-engine planes be a better choice?

6. Math 191 student Andrew Chi suggested last year in an email to me, "Perhaps with a few more restrictions, we would be able to prove a theorem that guarantees that the whole family is independent.

   Do you think there might be something like that?"
   Here is the best I could find (from Billingsley, *Probability and Measure*).For events $A_1, ...A_n$, consider the 2^n equations

$$\mathbb{P}(B_1 \cap ... \cap B_n) = \mathbb{P}(B_1)...\mathbb{P}(B_n)$$

   with $B_i = A_i$ or $B_i = A_i^c$ for each i.
   Show that $A_1, ...A_n$ are independent if all these equations hold.

7. Invent an example of identity (10) at the bottom of p. 29 in PRP based on three coin tosses, where A is "precisely 2 heads occur" and the events B_0, B_1, B_2 (lower index changed to 0 for convenience) refer to the number of heads in the last two tosses.