MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Assignment #11

Problems to be discussed in section on Monday, January 10: All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1, 3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 5.12, problem 1. This is just like an example done in lecture, but it takes some work to figure out all the right binomial coefficients!
2. Section 5.12, problem 4. The book's solution makes direct use of the probability mass function. Explain also how you can get the answer from the generating function for the geometric distribution.
3. Section 5.1, problem 2.
4. Section 5.1, problem 9. Invent a simple example for each case.
5. Section 5.1, problem 6. (binomial distribution with random $p$ is uniform!)
6. Section 5.2, problem 8. (Poisson distribution with exponential random parameter $\Lambda$ is geometric!)

Problems to be handed in on Thursday, January 13:

These are fairly straightforward problems designed to encourage you to take a closer look at some of the results proved in lecture.

1. An unfortunate premedical student must take a semester of calculus in college in order to apply to medical school, and he wants it to be the notorious gut course Math 4. Placement into Math 4 requires a score of precisely 6 out of 15 on the math placement test. There are two options for this placement test. Test A has three questions, on each of which the student has equal probability of getting any score from 0 to 5. Test B has five questions, on each of which the student has equal probability of getting any score from 0 to 3.
   (a) Write down the probability generating functions $G_A(s)$ and $G_B(s)$ for the placement tests, in the same style as problem 5.12.1 in the text.
   (b) By expanding and finding the coefficient of $s$ in each case, determine which leads to the higher probability for a score of precisely 6. If you want to check your answer, it is convenient that both probabilities are calculated by sets.exe, which you can download from the course Web site.

2. (a) Provide the missing details for the derivation of the generating function $Q(s, x)$ for the hypergeometric distribution on the middle of page 152 of PRP.

   (b) Obtain the solution to the original Munchkin problem (6 Munchkins, 2 of which are chocolate, with 3 placed in Thomas's bag) from the function $Q(s, x)$.

   (c) Use generating functions to calculate the variance for the hypergeometric distribution, which arises in "Munchkin problems." This calculation was left as an exercise by the authors.

3. An ambitious college graduate decides that before contemplating marriage he should do at least one job that makes him famous, one that makes him rich, and one that makes him happy. Every year he takes a new job. These jobs independently confer one attribute, fame, money, or happiness, each with probability $\frac{1}{3}$. Denote by $T_1$ the number of jobs that the graduate takes until he acquires one attribute, by $T_2$ the number of additional jobs that he takes until he acquires a second attribute, and by $T_3$ the number of additional jobs that he takes until he acquires the third and final attribute. The first of these random variables has a trivial distribution, and the other two each have a geometric distribution. The total time before the student is ready for marriage is $T = T_1 + T_2 + T_3$.

   Write down the probability generating functions for $T_1$, $T_2$, $T_3$, and $T$. Then write the generating function for $T$ as a sum of partial fractions, and thereby determine the probability mass function for $T$.

   This is a simplification of problem 5.2.9 in the text, whose solution (to put it mildly) is terse.

4. Solve the "Pedro at the plate" problem from assignment 6 by expanding the generating function for the "problem of the points,"

$$G(x, y) = \frac{y}{1-y}(1 - \frac{px}{1-qy})^{-1}$$

   in a power series and keeping just the term in $x^4y^3$. Your result will match up term-for-term with one of the approaches from Assignment 6, problem 1 – which one?

5. (This problem is from last year's final exam).

For centuries the young men of the tropical island of Octalia have played a secret gambling game in which they generate random decimal digits by rolling three octahedral dice, each with 0, 1, 2, and 3 on two of the eight faces, and summing the three numbers that appear. Donald Trump opens a casino on their island that features this game, and he then patents the game. The International Court of Justice rules that a probability mass function cannot be patented, nor can the concept of using octahedral dice, but that Trump's patent on the island's traditional mechanism for generating random digits with this mass function is valid.

Fortunately, the oldest son of the island's chief is skilled with generating functions and invents around Trump's patent.

Let the random variable $X$ be the roll of one of the dice, and let the random variable $Y$ be the sum of the rolls on three dice.

(a) Write down the generating functions $G_X(s)$ and $G_Y(s)$ for the random variables $X$ and $Y$.

(b) Suppose that a random variable $U$ has $\mathbb{P}(U = i) = p_i$ for each non-negative integer $i$. Derive the formula for the expectation of such a random variable $U$ in terms of its generating function $G_U(s)$, and apply this formula to determine the expectation of $Y$.

(c) By writing the generating function as a quotient and expanding in an infinite series, determine $\mathbb{P}(Y = 4)$.

(d) By factoring the generating function $G_Y(s)$ cleverly, invent a way of obtaining the same mass function as for random variable $Y$ by summing the rolls on two non-identical octahedral dice. For each of these dice, specify what numbers are used and on how many faces each appears. Use them to recheck your calculation of $\mathbb{P}(Y = 4)$.