

MATHEMATICS 191, FALL 2004  
MATHEMATICAL PROBABILITY  
Assignment #10

Problems to be discussed in section on Monday, December 13: All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1, 3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 4.14, problem 26. (making a triangle from a broken stick)
2. Section 4.13, problem 8. (area of random triangle similar to a given triangle)
3. Section 4.14, problem 8a. (another version of Bertrand)
4. Section 4.13, problem 9. (application of Crofton's method)
5. Section 4.13, problem 14. (choose 2 points inside a disk)
6. Section 4.14, problem 60. This is a lovely problem, but a tedious one. Explain all the steps, but don't take the time to evaluate the integrals.

Problems to be handed in on Thursday, December 16:

These are all relevant to the quiz on December 16. Be sure that you can use Crofton's method!

1. Two archeological teams both want to excavate along the same kilometer of road in Iraq. The local authorities lease a supercomputer from Halliburton and use it to generate four independent random variables  $X_1, X_2, X_3, X_4$ , all uniformly distributed in  $[0,1]$ . The first team is allowed to dig in  $[X_1, X_2]$ ; the second team is allowed to dig in  $[X_3, X_4]$ . Calculate the probability that their sites do not overlap, using two different methods, as follows:
  - (a) Let  $U = \max(X_1, X_2)$ . Find a distribution function  $F_U(u)$  and a density function  $f_U(u)$ . An easy way to get  $F_U(u)$  is to note that the point  $(X_1, X_2)$  would have to lie in a certain square if  $U \leq u$ .
  - (b) Let  $V = \min(X_3, X_4)$ . Find the probability  $\mathbb{P}(V > u)$ .
  - (c) By conditioning on the value of  $U$ , find  $\mathbb{P}(U \leq V)$ . Double this to get the answer.
  - (d) Alternatively, just note that if you arrange  $x_1, x_2, x_3$ , and  $x_4$ , all permutations are equally likely. Count the number of permutations that lead to no overlap.

2. Two points, chosen independently at random inside the unit square with uniform density, are taken as opposite corners of a rectangle with horizontal and vertical sides.
  - (a) What is the expectation of the area of the rectangle?
  - (b) What is the probability that two such rectangles, chosen independently, do not overlap? (You can reuse a result about nonoverlapping intervals from the preceding problem.)
  - (c) Using affine transformations, state generalizations of these results to parallelograms. The proof would be trivial, so you need not write it out.
3. There is a slightly simpler way of finishing the proof of (7) on p. 138 of PRP. Consider a triangle whose vertices are  $A = (0, 0)$ ,  $B = (x, 0)$ ,  $C = (0, 1)$ . Show that the same approach as in the text leads to the differential equation

$$\frac{dA}{dx} + \frac{3A}{x} = \frac{1}{6}$$

and solve this equation with boundary condition  $A(0) = 0$ , perhaps by noticing that  $x^3$  is an integrating factor.

4. A helicab driver hopes to pick up passengers along a road that extends from  $x = 0$  to  $x = 1$ . If the driver waits at position  $x = s$  and passengers arise at a random location  $X$  with uniform density along the road, determine  $\mathbb{E}((s - X)^2)$ . The result is a function of  $s$ . Sketch a graph of it, and then calculate its expectation under the assumption that the position of the helicab driver is also uniformly distributed in  $[0, 1]$ .
5. Confirm your calculation by computing an easy integral to determine  $\mathbb{E}((X - S)^2)$ , where  $X$  and  $S$  are uniformly and independently distributed in  $[0, 1]$ .
6. Now suppose that the helicab driver of the previous problem hopes to pick up passengers whose position is uniform within a square of side  $z$ .
  - (a) Using the result of the previous problem, show that if the helicopter is situated at random on the boundary of the square, the expectation of the square of the distance to a passenger is  $z^2/2$ .
  - (b) Let  $B(z)$  denote the expectation of the square of the distance from the helicopter to the passenger if both have a position that is independently and uniformly distributed within the square. Use Crofton's method to set up a differential equation for  $B(z)$ , and solve it subject to the boundary condition  $B(0) = 0$ .
  - (c) Confirm your solution by a much simpler calculation. You can set up a trivial 4-dimensional integral, or argue from the result of the previous problem, or look at problem 4.14.55 in the text.