Math 191 Notes, 2003 October 28

Random walks 2

Random Walk statistics

You will expect that most random walks will spend most of their time on one side (positive or negative).

The number of paths that never return to the origin (level 0) before step $2m$ is called $N_{\neq 0}(2m)$.

$$
\begin{aligned}
N_{\neq 0}(2m) &= 2\sum_{k=1}^{m}\binom{2m}{m+k}\underbrace{\frac{(m+k)-(m-k)}{2m}}_{\text{Ballot thm}} \\
&= 2\left[\sum_{k=1}^{m}\frac{(2m-1)!}{(m+k-1)!(m-k)!} - \sum_{k=1}^{m-1}\frac{(2m-1)!}{(m+k)!(m-k-1)!}\right] \\
&= 2\left[\sum_{k=1}^{m}\frac{(2m-1)!}{(m+k-1)!(m-k)!} - \sum_{j=2}^{m}\frac{(2m-1)!}{(m+j-1)!(m-j)!}\right] \\
&= \frac{(2m)!}{m!m!} = \binom{2m}{m}
\end{aligned}
$$

This is the number of ways of getting exactly half heads and half tails when we toss $2m$ coins.

Continue proof: look at a positive path:

$$N_{>0}(2m) = \frac{1}{2}N_{\neq 0}(2m) = \frac{1}{2}N_{\geq 0}(2m) = N_{\text{no overshoot final}}(2m)$$

These are all geometrically, not shown here. Look at written notes.

Arc sine laws

Assume symmetric random walk. Let define the following:

$$u_{2m} = \mathbb{P}(S_{2m} = 0) = \frac{1}{2^{2m}}\binom{2m}{m}$$

$$\alpha_{2m} = \mathbb{P}(\text{most recent return to level 0 was at time } 2k \text{ out of } 2m \text{ steps})$$

$$\alpha_{2m}(2k) = u_{2k}u_{2m-2k}$$

This is possible because the probability of a "zero" walk is the same as the probability of a "non-zero" walk when $p = 1/2$. A slightly harder one is:

$$\beta_{2m}(2k) = \mathbb{P}(\text{ } 2k \text{ of } 2m \text{ segments are in "positive territory"}) = u_{2k}u_{2m-2k}$$

The same law! Proof: Let $T = 2r$ be the time of the first return to the origin. Let the following:

$$f_{2r} = \mathbb{P}(\text{first return to origin is at step } 2r) = \frac{u_{2r}}{2r-1}$$

Note that if we condition on $2r$,

$$u_{2n} = \sum_{r=1}^{n} u_{2n-2r} f_{2r}$$

So then,

$$\beta_{2m}(2k) = \frac{1}{2}\sum_{r=1}^{k} f_{2r}\beta_{2m-2r}(2k-2r) + \frac{1}{2}\sum_{r=1}^{n-k} f_{2r}\beta_{2m-2r}(2k)$$

We will prove what we need by induction. Base case: $m = 1, \beta_2(0) = \frac{1}{2}$. Note that we can use all we know about u's[1]. $\beta_2(2) = \frac{1}{2} = u_2 u_0$. Inductive step: Assume

$$\beta_{2r} = \frac{1}{2}\sum_{r=1}^{k} f_{2r}u_{2k-2r}u_{2m-2k} + \frac{1}{2}\sum_{r=1}^{m-k} f_{2r}u_{2k}u_{2m-2r-2k}$$

$$= \frac{1}{2}u_{2m-2k}u_{2k} + \frac{1}{2}u_{2k}u_{2m-2k} = u_{2k}u_{2m-2k}$$

Proof by induction complete.

We want to describe this in terms of quartiles (is that the name)? For $m = 3, 2m = 6$, $\alpha_{2k}(2m) = u_{2m}u_{2m-2k}$.

[1] $u_0 = 1$, $u_2 = \frac{1}{2^2}\binom{2}{1} = \frac{1}{2}$