Math 191 Notes, 2003 October

Indpendence?

Independent X,Y means

$$\begin{aligned} PP(X \leq x, Y \leq y) &= PP(X \leq x)PP(Y \leq y) \\ PP(X = x, Y = y) &= PP(X = x)PP(Y = y) \end{aligned}$$

Example: Toss N coins (Poisson) get x heads, y tails. X = number of heads, Y = number of tails.

$$\begin{aligned} PP(X = x, Y = y) &= e^{-\lambda} \frac{\lambda^{(x+y)}}{(x+y)!} \frac{(x+y)!}{x!y!} p^x q^y \\ &= \left(\frac{e^{-p\lambda} \lambda^x p^x}{x!} \right) \left(\frac{e^{-q\lambda} \lambda^y q^y}{y!} \right) \end{aligned}$$

If $f(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for some λ, "X has a Poisson distribution."

Cauchy-Schwarz Inequality

Take random variables X and Y, look at $aX + bY$. That is a vector space of random variables. Is tehre any way we could assign a non-negative inner product whose square root might be interpreted as a length? Look at variance. Where $EE(X) = 0$, $EE(X^2)$, $EE(X)EE(Y) = EE(XY)$. Let $Z = \lambda X + Y$.

$$EE(Z^2) = EE(\lambda^2 X^2 + 2\lambda XY + Y^2) \geq 0$$

$$\lambda^2 EE(X^2) + 2\lambda EE(XY) + E(Y^2) \leq 0$$

Unles $Z = 0$, this quadratic has no real roots. Therefore,

$$(2EE(XY))^2 - 4EE(X^2)EE(Y^2) \leq 0$$

$$EE(XY) \leq \sqrt{EE(X^2)EE(Y^2)}$$

Now use $X - EE(X)$ and $Y - EE(Y)$

$$EE((X - EE(X))(Y - EE(Y))) \leq \sqrt{EE((X - EE(X))^2)EE((Y - E(Y))^2)}$$

$$cov(X, Y) \leq var(X0var(Y)$$

Given X, Y with expecation 0. We call the correlation, ρ as

$$\rho(X, Y) \equiv \frac{cov(X, Y)}{\sqrt{var(X)var(Y)}}$$

$$|\rho(X,Y)| \leq 1$$

with equality only if $\lambda X + Y = 0$.

Poisson

Random N (number of apps) is Poisson with parameter λ. Each application is successful independently with probability p.

- K is a random variable, the number of acceptances.
- Average number of applications $EE(N) = \lambda$
- Average number of acceptances $p\lambda$
- $EE(K|N=n) = pn$
- EE(N — K=k)

Now,

$$\begin{aligned} PP(N=n|K=k) &= \frac{PP(N=n, K=k)}{PP(K=k)} \\ &= \frac{PP(K=k|N=n)PP(N=n)}{PP(K=k)} \\ PP(N=n|K=k) &= \frac{\binom{n}{k}p^k q^{n-k} e^{-\lambda}\frac{\lambda^n}{n!}}{\sum_{n\geq k}\binom{n}{k}p^k q^{n-k} e^{-\lambda}\frac{\lambda^n}{n!}} \end{aligned}$$

Then, setting $n-k=i$,

$$\begin{aligned} EE(N|K=k) &= \sum nPP(N|K=k) \\ &= \frac{k\sum\frac{(q\lambda)^i}{i!} + \sum i\frac{(q\lambda)^i}{i!}}{\sum\frac{(q\lambda)^i}{i}} = k + q\lambda \end{aligned}$$