Math 191 Notes, 2003 October

Quiz Thursday: Combinatorial, Conditional, Properties of distribution functions, Theoretical

Discrete distribution

Mass function $f(x) = \mathbb{P}(X = x)$

$$\sum f(x_i) = 1$$

A couple of examples are the binomial distribution:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $x = 0, 1, \ldots, n$. The Poisson distribution is a limit of the binomial distribution as x grows very large:

$$f(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

where $k = 0, 1, 2, \ldots$

$$\sum_{k=0}^{\infty} f(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1$$

Euler Pi distribution:

$$\frac{1}{1^4} + \frac{1}{2^4} + \cdots = \frac{\pi^4}{90}$$

Independence of Random variables

Two random variables X and Y are independent if $PP(X = x, Y = y) = PP(X = x)PP(Y = y)$.

Toss a coin N times, where N is Poisson. What that means is that

$$PP(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

The expected value of the random variable is λ. Number of heads is X, the number of tails Y, are random variables. Because this is Poisson, they are independent.

$$PP(X = x, Y = y) = e^{-\lambda} \frac{\lambda^{x+y}}{(x+y)!} \frac{(x+y)!}{x!y!} p^x q^y$$

where $q = 1 - p$. This gives

$$= e^{-p\lambda} \frac{(p\lambda)^x}{x!} \frac{(q\lambda)^y}{y!}$$

Expectation

Intuitive for most people. Expectation of X:

$$\mathbb{E}(X) = \sum x\mathbb{P}(X = x)$$

This is NOT the formula most frequently used for this, but it is the one and only definition. **"Law of unconscious statistician"** The way I do it, not formally correct.

Rearranging stuff doesn't work on the following integral:

$$\begin{aligned}\sum_{n=1}^{2}\sum_{k=0}^{\infty}\frac{1}{n}\frac{(-1)^k}{k+1} &= 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \cdots \\ &=? \ 1/2 - 1/4 + 1/6 - 1/8 + \cdots\end{aligned}$$

If we add these together we get the original sequence! What happens is that if we don't use the minus signs, then the series is divergent. The series is only conditionally convergent as opposed to absolutely convergent.

Application to economics. Well-known stupid joke: A company is interviewing for CFO – what is 2+2 – what do you want it to be, you're hired.
Revenues of \$1M, \$1/3M, \$1/5M, $\cdots$
Expenses of \$1/2M, \$1/4M, $\cdots$

Variance

$$var(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

$$\mu = \mathbb{E}(X)$$

$$var(X) = \mathbb{E}((X - \mu)^2)$$

$$var(X) = \sigma^2, \sigma = \text{ "standard deviation"}$$

$$\begin{aligned}var(X) &= \sum (x-\mu)^2 f(x) \\ &= \sum x^2 f(x) - \sum 2\mu x f(x) + \sum \mu^2 f(x) \\ &= \mathbb{E}(X)^2 - 2\mu\mathbb{E}(x) + \mu^2 \\ &= \mathbb{E}(X^2) - \mu^2 \\ &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2\end{aligned}$$

Uncorrelated

Random variables X and Y are said to be **uncorrelated** if

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

NOT THE SAME AS INDEPENDENCE!!!!!

Independence $\Rightarrow$ uncorrelated

$$\mathbb{E}(XY) = \sum xyPP(X = x)PP(Y = y) = \sum xPP(X = x) \sum yPP(Y = y)$$

works because of independence and because of absolute convergence.

Uncorrelated $\not\Rightarrow$ independence Here is an example: students take two independent tests which you either pass or fail, all you can do is guess and each one is 50% chance of getting things right. Pass = +1, Fail = -1.
"Achievement": $X + Y$, X = score on #1, Y = score on #2
"Improvement": $Y - X$
If students do this at random by flipping coins, then the two are uncorrelated. All four outcomes happen with equal probability, and the achievement scores are 2,0,0,-2. The improvement scores are 0,-2,2,0. The expectation of achievement is 0, the expectation of improvement is 0. Expectation of the product is 0.

Variances of uncorrelated events

If X and Y are uncorrelated, then

$$var(X + Y) = var(X) + var(Y)$$

Proof:

$$\begin{aligned} var(X + Y) &= \mathbb{E}((X + Y)^2) - (\mathbb{E}(X + Y))^2 \\ &= \mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - (\mathbb{E}(x))^2 - 2\mathbb{E}(X)(Y) - (\mathbb{E}(Y))^2 \\ &= var(X) + var(Y) \end{aligned}$$

In fact,

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

But variance is NOT a linear operator.

$$var(aX) = \mathbb{E}(a^2X^2) - a^2(\mathbb{E}(X))^2 = a^2var(X)$$