MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Outline #2 (Sets and Probability)

References: Grimmett & Stirzaker, Sections 1.1 to 1.3 and 1.7
Web pages on Poker and Bridge from www.durangobill.com (attached)

1. Derive formulas for the intersection and difference of sets A and B in terms only of union and complement. Illustrate the formulas using Venn diagrams and using the program set.exe.

2. State G&S's definition of a σ-field and explain why the closure requirements that they impose are sufficient to prove closure under difference and intersection. Give a couple of trivial examples of σ-fields(G&S, p.3)

3. Define "probability measure," making it clear what additional requirement is imposed on a measure if it is to be a probability measure. Define "probability space", and illustrate the definition by describing the probability space associated with rolling a single die.

4. Prove that for a probability measure $\mathbb{P}$,
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
and illustrate this theorem with a diagram where S is the set of points in a square region of the blackboard.
Prove that for a probability measure $\mathbb{P}$,
   - $\mathbb{P}(A \cup B) <= \mathbb{P}(A) + \mathbb{P}(B)$
   - $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$
   - $\mathbb{P}(A) \leq \mathbb{P}(B)$ if $A \subseteq B$.

   (G&S p. 6)

5. State and prove Lemma 5 on page 6 of G&S, and explain why this lemma would be inappropriate if the definition of a probability measure involved only finite unions and inadequate if the definition allowed uncountable unions.

6. Suppose that the sample space Ω consists of the integers 2 through 12 inclusive. These are the totals that can be realized by tossing two dice. Invent a number of subsets of Ω, then make statements about these subsets in the language of probability and events and explain the basis for each statement in terms of sets. A simple example: "The probability that the die roll is odd is no greater than the probability that it is between 3 and 11." Set explanation: The set of odd numbers between 2 and 12 is a subset of the numbers between 3 and 11. Use Table 1.1 on page 3 of G&S as a guide.

7. Using the principles listed at the start of section 1.7 in G&S, explain how to answer the following questions:

   (a) What is the probability of getting exactly two heads when a fair coin for which $\mathbb{P}(h) = \frac{1}{2}$ is tossed three times?

   (b) What is the probability, when two cards are drawn at random from a single deck of cards, that at least one of them is a spade?

   (c) What is the probability of getting a total of either 7 or 11 when tossing two "unloaded" dice?

   In each of the next three items, refer to these principles of counting.

   - Principle 1: Multiply for sequential counting. If we are forming a set of n-tuples where there are k_1 choices for the first element x_1, k_2 for x_2, and so on, the number of n-tuples $x_1x_2...x_n$ in the set is $k_1k_2...k_n$
   - Principle 2: Divide to correct systematic overcounting. If M different n-tuples all correspond to the same element in our sample space, count the n-tuples and then divide by the number of times M that each was overcounted.
   - Principle 3: Divide and Conquer. To count the number of elements in a union of disjoint subsets, count each subset and sum the results.
   - Principle 4: Subtract off special cases. To count the number of elements in a difference $A \setminus B$ where $B \subset A$, count each set and take the difference.

   Classic example of principles 1 and 2: counting the number of k-element subsets of an n-element set.

   First count the k-tuples whose elements are all different:

   $$n(n-1)(n-2)...(n-k+1).$$

   Then divide by $k!$ to correct for overcounting. The result is the familiar "combinations" formula

   $$fracn!k!(n-k)!$$

8. In the carnival game "Chuck-A-Luck," you pick a number between 1 and 6. Three fair dice are tossed, and you win if your chosen number appears on one on more dice.

   - Show that your probability of winning is less than 1/2.
   - Determine the probability that your chosen number will appear on 0, 1, 2, or all 3 dice.

- Show that if you pay 1 dollar to play the game and receive 2 dollars for each occurrence of your chosen number, then the game is fair.

Use principle 4, where A is the set Ω of all possible rolls and B is the set of winning rolls, so $P(B) = 1 - P(B')$ where B' is the set of losing rolls.

The number of losing rolls is 5 x 5 x 5 = 125.
The total number of rolls is 6 x 6 x 6 = 216
So there are 216-125 = 91 winning rolls, and your chance of winning is 91/256.

Sanity check:

- 1 roll with 3 of your number.
- 15 rolls with 2 of your number (3 places for the other number x 5 choices for it).
- 75 rolls with 1 of your number (3 places for it, and 5 x 5 pairs of other numbers for the other two dice)
  $3 \times 1 + 15 \times 2 + 75 \times 1 = 108$, which is half of 216.

For the following topics, the easiest way to get numerical answers is to use Mathematica. Any Harvard undergraduate can download a copy for free. For details, go to http://icg.harvard.edu/software/mathematica and follow instructions.

9. Count the number of ways to get each of the following types of 5-card poker hands, using a deck of 52 cards with 4 cards of each of 13 ranks.

   - 4 of a kind(four cards of one rank, the fifth of a different rank)
   - a full house(three cards of one rank, two of another)
   - 3 of a kind(three cards of one rank, two others of different ranks)

   Solutions:

   - How many distinct 5-card poker hands can be dealt from a 52-card deck?
     Answer: Select 5 cards sequentially: the number of ways to do this is $52 \times 51 \times 50 \times 49 \times 48$
     But this generates each distinct hand $5! = 120$ times.
     So the number of distinct hands is $\frac{52!}{47!5!}$
   - How many distinct ways are there to get 4 of a kind?
     13 choices for the rank of the 4, 48 choices for the other card.
   - A full house (3 of one rank, 2 of another)
     13 choices for the first rank (with 3 cards), 4 for the suit that is missing.
     12 choices for the second rank (with 2 cards), 6 for the pair of suits that have the second rank.

- Three of a kind (3 of one rank, the other two do not match)
  13 choices for the first rank (with 3 cards), 4 for the suit that is missing.
  48 choices for the fourth card
  44 choices for the fifth card (because 2 ranks are now excluded)

10. Bridge problems

    Count the number of bridge hands with 6 spades, 4 hearts, 2 diamonds, and 1 club.

    - Count the number of bridge hands with 6-4-2-1 suit distribution (6 cards in the longest suit, 4 in the second-longest, 2 in the third-longest)
    - Count the number of bridge hands with 4-4-3-2 or 4-3-3-3 suit distribution, and show that the former has a higher probability.

Solution

- How many ways are there to select 6 cards from the 13 spades?

$$\frac{13!}{7!\,6!}$$

- How many distinct hands have 6 spades, 4 hearts, 2 diamonds, 1 club?
  Apply the same analysis to each suit in turn:

$$\frac{13!}{7!\,6!}\frac{13!}{9!\,4!}\frac{13!}{11!\,2!}\frac{13!}{12!\,1!}$$

- How many distinct hands have a 6-4-2-1 distribution?
  Multiply the preceding number by the number of ways to choose the 6-card suit, the 4-card suit, etc, which is 24.
- How many distinct hands have a 4-4-3-2 distribution?
  Start with

$$\frac{13!}{9!\,4!}\frac{13!}{9!\,4!}\frac{13!}{10!\,3!}\frac{13!}{11!\,2!}$$

  Now there are 4 ways to select the 2-card suit followed by 3 ways to select the 3-card suit, so multiply by 12.
  (Alternative: multiply by 24 as above, but divide by 2 to correct for overcounting)