MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Outline #6 (Random Walks)

Reference: G&S, sections 3.9 and 3.10

1. As an example of a simple random walk, consider a computer game where the player starts at level $a \geq 0$ and, at every step, either moves up one level with probability p or moves down one level with probability $1 - p$. The game is won if the player reaches level N and lost if the player reaches level 0. Express symbolically, and prove, the following properties:

   - It is spatially homogeneous.
   - It is temporally homogeneous.
   - It has the Markov property.

2. Let p_k denote the probability that the player starts in level k and eventually loses the game. Set up and solve a recurrence relation to determine p_k, first in the general case where $p \neq \frac{1}{2}$ and then in the special case where $p = \frac{1}{2}$.

3. Let D_k denote the expected number of steps after which the player, starting in level k, wins or loses the game. Set up and solve a recurrence relation for D_k, and solve it for the special case where $p = \frac{1}{2}$. Discuss what happens in the limit $N \to \infty$.

4. Allow "negative levels," and use the reflection principle to count the number of paths $N_n^0(a, b)$ that start at level a and end at level b while passing through level 0. Then state and prove the ballot theorem, which states that the fraction of paths going from level 0 to level b in n steps that never pass through level 0 again equals $\frac{b}{n}$.

5. State the "principle of reversal", and use it to show that in the special case where there are negative levels, $p = \frac{1}{2}$ and the player starts in level 0, the mean number of times the player is in level b before losing the game is 1.

6. Again consider the special case where there are negative levels, $p = \frac{1}{2}$ and the player starts in level 0. Show that, if T_{2n} denotes the last time before step $2n$ that the player was in level 0 and $0 < x < 1$, then $\mathbb{P}(T_{2n} \leq 2xn)$ is approximately $\frac{2}{\pi} \arcsin x$.

7. Yet again consider the special case where there are negative levels, $p = \frac{1}{2}$ and the player starts in level 0. Show that the fraction x of the time that the player spends in positive levels during the first $2N$ turns is also approximately $\frac{2}{\pi} \arcsin x$.