MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Outline #1 (Countability and Uncountability)

Reference: Apostol, Calculus, Vol. 2, section 13.19 (attached). This is part of the course pack.

1. Functions from one set to another.

   Explain the two requirements for two sets to be in one-to-one correspondence. Illustrate these requirements (and how they can fail) for the case of finite sets with just four or five elements. Then show that the infinite set of even positive integers is in one-to-one correspondence with the set of positive integers.

   Here is some useful terminology that relates to the definition on p. 501 of Apostol. A function that satisfies (a) is called "onto" or "surjective." A function that satisfies (b) is called "one-to-one" or "injective." A function that satisfies both (a) and (b) is called "bijective" or "invertible." Two sets that are in one-to-one correspondence are called "equivalent."

   Notes: Sets A and B are in 1-to-1 correspondence if there exists a function
   $f : A \longrightarrow B$ such that

   - The range of f is all of B (f is surjective)
   - If x and y are distinct elements of $A, f(x)$ and $f(y)$ are distinct elements of B (f is injective).
     f is also surjective, and hence bijective.

   Such a function f has an inverse g, and the sets A and B are called equivalent: $A \sim B$ and $B \sim A$

   Example: the set of even positive integers is equivalent to the set of all positive integers

   A set that is equivalent to $\{1, 2, ...n\}$ is a finite set with n elements.

2. Countably infinite sets.

   Define the terms "countably infinite set" and "countable set." Then use this definition to show that the set of all integers (positive, negative, or zero) is countably infinite. Also show directly from the definition (without using other properties as Apostol does in Example 3 on p. 503) that the collection of all two-element subsets of the positive integers is countably infinite, as is the collection of all three-element subsets of the positive integers.

In a similar way, show that the set of positive rational numbers is countably infinite. This is a little bit tricky if you want to set up a 1-to-1 correspondence. To show that the rationals are countable (as opposed to countably infinite) you would need only to exhibit a function from the positive integers to the set that is onto (surjective). This would leave open the possibility that the set is finite.

Notes: A set that is equivalent to {1,2,3...} is a countably infinite set.
A computer version of this concept: Imagine a function that produces an infinite sequence of values, all in set B. If it produces each element of B once and only once, then B is countably infinite. If, given any supposed such program, you can invent an element of B that the function can never produce, then B is uncountable.

Properties: proofs are left as exercises in Apostol but are included below:

- a Every subset of a countably infinite set is countable
- b The intersection of ANY collection of countable sets is countable
- c The union of a COUNTABLE collection of countable sets is countable
- d The Cartesian product of a FINITE number of countable sets is countable

Doing the examples:

- The family of 2-element subsets of the positive integers is countable

  Proof: Alternative 1: Arrange the sets in order of increasing larger element, so the computer program produces the bijection
  1: $\{1, 2\}$
  2: $\{1, 3\}$
  3: $\{2, 3\}$
  4: $\{1, 4\}$
  5: $\{2, 4\}$
  and so on. Clearly any 2-element set will appear once and only once

  Alternative 2: Arrange the sets in order of increasing sum of elements, so the computer program produces the bijection
  1: $\{1, 2\}$
  2: $\{1, 3\}$
  3: $\{2, 3\}$
  4: $\{1, 4\}$
  5: $\{2, 3\}$
  6: $\{1, 5\}$
  7: $\{2, 4\}$
  8: $\{1, 6\}$
  9: $\{2, 5\}$
  10:$\{3, 4\}$

and so on. Clearly any 2-element set will appear once and only once.

- The set of rational numbers is countable:
  Proof: Use the "sum" approach (alternative 2 above) to generate all ordered pairs {numerator, denominator}. Omit any that lead to fractions that are not in lowest terms.
  1: 1/1
  2: 1/2
  3: 2/1
  4: 1/3
  5: 3/1 (note – 2/2 was omitted)
  6: 1/4
  7: 2/3
  8: 3/2
  9: 4/1
  10: 1/5
  11: 5/1 (note: 2/4, 3/3, and 4/2 were all omitted)
  and so on.

The Windows application enum.exe (available on the course Web site and in class) is a great way to make the definition vivid to your audience. For two-element subsets, challenge everyone to choose and write down such a set before you start up the program. Then show that a chosen set will show up once and only once on your list as the program runs and that you could have predicted exactly where on the list it would appear.

3. Subsets and intersections.

   Present the solutions to exercises 3 and 4 on p. 505 of Apostol, thereby establishing properties (a) and (b) on p. 502. As an example, you might show that the set of prime numbers is countable. You do not need to show that it is countably infinite, just countable!

   In dealing with the finite case, prove that every subset of a finite set is finite. Then you can state the obvious but useful corollary that if a set has a countably infinite subset, it is infinite.

   If Apostol's hint for exercise 4 is not quite enough, just remember that the intersection of a set A with any other set B must be a subset of A.

4. Cartesian products and unions.

   Present the solution to exercise 5 on p. 505 of Apostol, thereby establishing property (d) on p. 502.

   The inductive proof that Apostol suggests is a nice way to proceed, but you could also just extend the prime-number trick. For example, to show that the set of all sequences of 5 positive integers like (m, n, r, s, t) is countable, let $f(m, n, r, s, t) = 2^m 3^n 5^r 7^s 11^t$.

Next present the solution to exercise 6 on p. 505. This shows most of property (c) on p. 502, namely that the union of a countable collection of disjoint countable sets is countable. The extension to sets that are not disjoint is unenlightening.

Finally, present examples 3 and 4 on p. 503.

Here is an alternative direct proof for 3, "The family of n-element subsets of the positive integers is countable." Enumerate all n-element sets whose sum is 1, then 2, then 3, then 4, etc. This is a countable union of finite sets, hence countable.

The proof for 4, "The collection of all FINITE subsets of a countable set is countable", uses property (c). This is a countable union of countable sets, one for each value of n.

5. Uncountability of the real numbers.

   Present example 6 on p. 505 of Apostol. This is the simplest and best-known form of Cantor's ingenious argument. Here is a nice way of looking at it. Suppose that someone claims to have written a computer program that enumerates all the real numbers between 0 and 1 in a list, which might begin like this:

   1. $\pi - 3$
   2. $3/7$
   3. $\sqrt{2} - 1$
   4. ....

   In terms of decimals the list will look like this:

   1. .141592....
   2. .428571428...
   3. .414213562....

   You construct a number as proposed by Apostol. For the given example it will begin .211.... Since it differs in at least one decimal place from each number on the list, it cannot appear anywhere on the list; so the author of the program is making an incorrect claim.

   This is how all proofs of uncountability work. Assume that the set is countable, and show that this assumption leads to a contradiction.

6. Uncountability in general.

   Present Example 5 on pp. 503-504 of Apostol. The key result is that a function from a set A to the collection of all subsets of that set cannot be onto (surjective). The way you prove this is to assume the existence of a surjective function f and to show that this assumption leads to a contradiction. Before presenting Apostol's proof it is worth doing the example of a finite set, say with four elements. In this case the result is obvious, since there are 16 different subsets of {1,2,3,4}, but what you should do is to ask your audience to choose $f(1), f(2), f(3)$, and $f(4)$.

Suppose you end up with $f(1) = \{1, 2\}$
$f(2) = \{3\}$
$f(3) = \{1,2,4\}$
$f(4) = \{1,2,3,4\}$

Now you note that $f(1)$ includes 1 and $f(4)$ includes 4, but $f(2)$ fails to include 2 and $f(3)$ fails to include 3. So you construct the set {2,3}, which cannot be in the range of f. The fact that you can always do this shows that your audience cannot create a surjective f. This is Apostol's argument, and it works for a countably infinite set as well as for a finite set.

Here is a summary of Apostol's argument.

The collection of ALL subsets of a countably infinite set is UNcountable

Proof (by contradiction): Imagine a computer program that generates a sequence of all subsets of the positive integers: Sometimes the nth element in this sequence will include the integer n; sometime it will not. Let B be the set of all integers n for which the program generates a subset that does not include n.
Assume that the function generates B as the bth element in the sequence. Either b is an element of B, or it is not.
Now show that either alternative implies its opposite!

First, assume that b is an element of B.
But the definition of B is that b is not an element of B, so if b is an element of B then b is not an element of B

Second, assume that b is not an element of B. But then b satisfies the criterion for membership in B, so if b is not an element of B then b is an element of B. So either assumption about b leads to a contradiction. Thus no such b can exist, and B is uncountable.

7. Borel's normal number theorem (carelessly stated and not proved).

   A number is called "normal" if, when it is expanded in any number system, all the digits occur with equal frequency. If π is normal, all the digits 0-9 should show up equally often. Experimental results for 1.24 trillion digits are consistent with this hypothesis. Argue that the following are equivalent:

   - Flip a fair coin infinitely many times.
   - Choose a real number between 0 and 1 at random.

and that non-normal random real numbers should be highly unusual, even though there are uncountably many of them.

8. The Schwarzkopf attack.

   This is a highly contrived example to show that subtle mathematical issues can arise out of what look like realistic problems.

   Your task is to defend 1 mile of border against an attack masterminded by Gen. Norman Schwarzkopf, who achieved fame for his surprise move from Saudi Arabia into Iraq during the 1991 Gulf War. Fortunately, a spy has captured the orders to the attacking commander. Here they are.

   0. At noon, take up a position at $x = \frac{1}{2}$ miles.
   1. Toss a coin. If it comes up heads, move $\frac{1}{3}$ of a mile east to $x = \frac{5}{6}$ miles. If it comes up tails, move $\frac{1}{3}$ of a mile west to $x = \frac{1}{6}$ miles. This can be accomplished in half an hour.
   2. Toss a coin. If it comes up heads, move $\frac{1}{9}$ of a mile east. If it comes up tails, move $\frac{1}{9}$ of a mile west. This can be accomplished in a quarter of an hour.

      ....

   n. Toss a coin. If it comes up heads, move $\frac{1}{3^n}$ of a mile east. If it comes up tails, move $\frac{1}{3^n}$ of a mile west. This can be accomplished in $\frac{1}{2^n}$ of an hour.

      ....

   final. At 1 PM, attack!

   (a) Show that the infinite number of steps implied by the orders require 1 hour.

   (b) Show the the sum of the lengths of all the intervals where the attacking commander cannot be located is 1 mile, so that a defending unit that takes up a random position is almost certain to be in the wrong place.

   (c) Show that at 1 PM, the set of points where the attacking commander can be located is uncountably infinite. (Hint: Express the position x as a fraction in a base-3 number system. For example .202 means $\frac{2}{3} + \frac{2}{27}$, the position resulting from heads-tails-heads. There is now an obvious 1-to-1 correspondence with the real numbers between 0 and 1, which are uncountable.)

   (d) Let $F(x)$ denote the probability that the attack occurs to the west of position x. Sketch a graph of $F(x)$, and argue that its derivative is zero almost everywhere but is undefined at uncountably many points.