MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Outline #3 (Conditional Probability)

Reference: G&S, sections 1.4, 1.5 and 1.7

1. Define conditional probability (G&S, p.9) and use sets.exe to show examples of how to calculate it.

2. State and prove Lemma 4 on p. 10 of G&S,

$$\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)$$

   Use this lemma and the formula for conditional probability to analyze the "bearded man" problem in the notes or a similar example of your own invention.

3. Use conditional probability to analyze example 3 on page 9 of G&S.

4. Describe the "Monty Hall problem" and analyze it in terms of conditional probability. This is equivalent to example 7 on page 9 of G&S.

5. Define independent events (G&S, p. 13) and illustrate the concept using sets.exe.

6. Explain how to create independent events by means of a "compound experiment," for example a die roll followed by a coin flip. Use sets.exe to illustrate events that are independent whenever "Random 4 × 4" is used.

7. Define Bernoulli trials (see the notes), and show that the probability of exactly k successes in n Bernoulli trials is

$$\frac{n!}{k!(n-k)!}p^k q^{n-k}$$

   where p is the probability of success in one trial.

8. Give an example of a simple probability problem that leads to a countably infinite sample space (for example, rolling a die over and over until a "6" appears), and show that it leads to a convergent infinite series that you know how to sum.

9. Give an example of Simpson's paradox and state the paradox in general terms (G&S, p. 19).

Notes:

- Conditional probability: Suppose that A and B are events (subsets) of a sample space Ω. If we know that event B has occurred, what is then the probability that A has also occurred?

  Example: I roll two dice and note that the sum is six. What is the probability that at least one 2 is showing?
  Answer: there are 5 pairs in set B: (1,5), (2,4), (3,3), (4,2), and (5,1)
  Two of them belong to set A so the probability is $\frac{2}{5}$

  at least one 3 is showing?
  Answer: there are 5 pairs in set B: (1,5), (2,4), (3,3), (4,2), and (5,1)
  Only one of them belongs to set A so the probability is $\frac{1}{5}$

  Definition of conditional probability: Suppose that $P(B) > 0$.
  Then $P(A|B)$ "the probability of A, given B," is

  $P(A|B) = P(A \cap B)/P(B)$

  Many simple problems in conditional probability can be analyzed by arranging the probabilities into a rectangular array where the rows and columns correspond to mutually exclusive events and each cell gives the probability for the intersection of the "row" and "column" events.

- Example: the "bearded man" problem:

  At an airport with severe terrorism problems, security personnel have established the following:
  Of male passengers with explosives in their shoes, 60% have beards.
  Of male passengers with no explosives in their shoes, 5% have beards.
  20% of male passengers have explosives in their shoes.
  What is the probability that a bearded male passenger has explosives in his shoes?

  Event A is "explosive shoes" while event A^c is "non-explosive shoes"
  Event B is "bearded" while event B^c is "clean-shaven"

  From the given information we can make a table

	B	B^c
A	.12	.08
A^c	.04	.76
Sum	.16	84

  Calculate $P(B)$ by using the lemma on page 10 of G&S.

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = .05 \times .8 + .60 \times .2 = .16$$

  So $P(A \cap B) = .12, P(B) = .16$, and

$$P(A|B) = P(A \cap B)/P(B) = .12/.16 = 3/4.$$

- The most famous probability problem (the Monty Hall problem)

  In a computer game show, there are three doors. Behind one door, chosen at random, is a car C. Behind each of the other two doors is a goat. There is a small goat $G1$ and a large goat $G2$. You select a door. Before revealing whether you have won the car, host Monty Hall, who knows where the car is located, opens a door that you have not chosen and that does not have the car behind it, revealing a goat. If he has a choice, he reveals the smaller goat. He then asks if you would like to switch your choice to the remaining unopened door.

  Suppose (without loss of generality) you pick door 1. There are six ways to arrange the car and the two goats behind the three doors

  Here are the six 6 equally likely possibilities, along with Monty Hall's action in each case.

Case	Door 1	Door 2	Door 3	Monty opens
1	C	G1	G2	door 2
2	C	G2	G1	door 3
3	G1	C	G2	door 3
4	G2	C	G1	door 3
5	G1	G2	C	door 2
6	G2	G1	C	door 2

  Event A is "the car is behind door 1"
  Event B is "Monty Hall opened door 2" So $P(A \cap B) = 1/6$ (case 1 only)
  $P(B) = 1/2$ (cases 1, 5, and 6)

  So $P(A|B) = P(A^B)/P(B) = 1/3, P(C|B) = 2/3$, and you double your chances of winning by switching to door 3!

  A subtle variant: Monty flips a coin to choose a door (not yours) at random and open it. This means that you may discover that you have lost. Now the 6 possibilities are

  Car behind 1, heads, opens 2
  Car behind 1, tails, opens 3
  Car behind 2, heads, opens 2
  Car behind 2, tails, opens 3
  Car behind 3, heads, opens 2
  Car behind 3, tails, opens 3

  Suppose that event B is "he opens 2, and there is no car behind it."

  So $P(A \cap B)$ is 1/6 from Car behind 1, heads, opens 2

  $P(B)$ is 2/6 from
  Car behind 1, heads, opens 2
  Car behind 3, heads, opens 2

Now $P(A|B)$ is 1/2 and there is no advantage to switching.

- Independence

  Two events A and B are called independent if

  $P(A \cap B) = P(A)P(B)$

  Alternative view: In this case

  $$P(A|B) = P(A \cap B)/P(B) = P(A)P(B)/P(B) = P(A)$$

  so that the conditional probability of A is unaffected by knowledge of B.

  Simple example:
  $A =$ "Drawing a heart"
  $B =$ "Drawing an ace"
  Independent - easy argument on G&S p. 13.

  $A =$ "Drawing two hearts"
  $B =$ "Drawing two aces"
  Not independent since $P(A \cap B) = 0, P(A) > 0$

  Slightly more complicated:
  $A =$ two fair dice are rolled and the numbers are the same.
  $P(A) = 1/6$
  $B =$ "sum of the numbers on the 2 dice is 8"

  There are 5 possibilities: (2,6) (3,5) (4,4) (5,3) (6,2) So $P(A \cap B) = 1/36$ and $P(B) = 5/36$. Thus $P(A|B) = 1/5$ and the events are not independent.

  For three events to be independent it is required that $P(A \cap B \cap C) = P(A)P(B)P(C)$

  Simple but subtle example (modified from example 2 on p. 13 of G&S)

  A computer program selects at random from the strings "a," "b," "c," "abc" Event A: the chosen string contains an "a" $P(A) = 1/2$ Event B: the chosen string contains a "b" $P(B) = 1/2$ Event C: the chosen string contains a "c" $P(C) = 1/2$

  Event AB: the chosen string contains an "a" and a "b: $P(AB) = 1/4 = P(A)P(B)$
  So the events are "pairwise independent."

  Event ABC: the chosen string contains an "a" , a "b", and a "c":
  $P(ABC) = 1/4$ but $P(A)P(B)P(C) = 1/8$

  In other words, once I know that the event $A \cap B$ has occurred, the conditional probability of event C is 1

- Compound Experiments
  A compound experiment might consist of a die roll followed by a coin flip. In such a case we can construct independent events in a very general manner.

  We assume that we know probabilities $P_1(x)$ for the first experiment (the die roll) even if the die is loaded. We assume that we know probabilities $P_2(y)$ for the second experiment (the coin flip) even if the coin is not true.

  The outcome of the compound experiment is a pair (x, y) e.g.)4, heads) It is reasonable to define $P(x, y) = P_1(x)P_2(y)$ e.g. 1/12 for each outcome if die and coin are fair. Note: with this formula, $P(x, y)$ is positive, and the sum over all possible outcomes is 1.

  Now we have to show that the condition for independence is satisfied:

  $$P(A \cap B) = P_1(A)P_2(B)$$

  This is tricky, since events A, B, and $A \cap B$ all have to be subsets of the same set.

  Event A: result of first experiment is in C_1, result of experiment 2 is anything. $P(A) = P_1(C_1)$

  Event B: result of second experiment is in C_2, result of experiment 1 is anything. $P(B) = P_2(C_2)$

  Event $A \cap B$: result of first experiment is in C_1, result of second experiment is in C_2.

  Then from the definition $P(x, y) = P_1(x)P_2(y)$ it follows that $P(A \cap B)$ $= P_1(A)P_2(B)$.

  An illustration is in sets.exe, the "Random 4 x 4" probabilities. It works this way:
  Generate 4 random numbers to get $P_1(q)$ for the quotient q. (values 0 - 3)
  Generate 4 more random numbers to get $P_2(r)$ for the quotient r. (values 0 - 3)

  Assign to each number $0 \leq x \leq 15$
  $P(x) = P_1(x/4)P_2(x \mod 4)$ Then, no matter what the random numbers, a "remainder event" like $x \mod 4 = 1$ is independent of a "quotient event" like $x \leq 7$.

- Bernoulli trials:
  Carry out a sequence of identical independent experiments. Select an event S (the same for each experiment) that you call "success." Call its

probability p. The complementary event is "failure" F: its probability is $q = 1 - p$.

Theorem: The probability of exactly k successes in n Bernoulli trials is

$$\frac{n!}{k!(n-k)!} p^k q^{n-k}$$

Proof:

The probability of success in k specified trials (for example, the first k) and failure in the remaining $n - k$ trials is

$$p^k q^{n-k}$$

The number of distinct ways of choosing k trials with success is

$$\frac{n!}{k!(n-k)!}$$

The product of the two gives the desired probability of k successes.

Example: What is the probability of getting exactly 2 fives when you roll a die 6 times?
Answer: The probability of getting five on 2 specified rolls, not-5 on the others, is

$$(\frac{1}{6})^2 \times (\frac{5}{6})^4$$

The number of ways of selecting 2 rolls from 6 is $\frac{6 \times 5}{2} = 15$. So the probability is

$$\frac{15 \times 5^2}{6^6}$$

- Probability for countably infinite sets
  Imagine a countable collection of disjoint sets. We assume that the union of this (infinite number of) sets is in the σ-field. Then if $P(A_1 + A_2 + A_3 + ...) = P(A_1) + P(A_2) + P(A_3) + ...$ we say that P is countably additive.

  We can specify P by giving its (non-negative) value $P(x)$ for each 1-element subset x. Of course we insist that $P(\Omega) = 1$ for the (countably infinite) universal set Ω.

  Then, since A may contain infinitely many elements.

  $P(A) = P(x_1) + P(x_2) + P(x_3) + ...$
  may be the sum of an infinite series, guaranteed to converge since its terms are non-negative and the sum cannot exceed 1.

  Simple example: Keep rolling a die until you roll a 6. What is the probability that this happens on the nth roll?

Answer: First roll: $P_1 = \frac{1}{6}$
Second roll: $P_2 = (\frac{5}{6})(\frac{1}{6})$
$n+1$st roll: $P_{n+1} = (\frac{5}{6})^n \frac{1}{6}$

The sum of all the probabilities is

$\frac{1}{6}(1 + \frac{5}{6} + (\frac{5}{6})^2 + (\frac{5}{6})^3 + ...) = \frac{\frac{1}{6}}{1-\frac{5}{6}} = 1$