Math 191 Notes, 2003 December 9

Quiz Thursday: 2 questions: 1. Here are two random variables X, Y, here are two other variables U, V in terms of X and Y. Give me all the characteristics, blah. 2. Crofton's Method. (Prepare geometric stuff!)

Review Wednesday night.

3 Old friends by Generating Functions

Matching Problem: 2 ways

Strategy: take some sort of recurrence that you can write down, multiply by some appropriate power of s, sum and take great care with the terms $n = 0$ and $n = 1$.

Someone is stuffing the usual n envelopes. The first one is wrong with probability $(n - 1)/n$. Without loss of generality, assume letter 1 gets stuffed into envelope 2. Then look at the remaining $n - 1$ letters. Consider envelope 1 (in the n case) to be equivalent to envelope 2 (in the $n - 1$ case). Then, we have:

$$P_n = \left(\frac{n-1}{n}\right)\left[P_{n-1} + \frac{1}{n-1}P_{n-2}\right]$$

$$nP_n = (n-1)P_{n-1} = P_{n-2}$$

Now let's convert this sequence of numbers into an analytic function. (Discussion about the countable/uncountable amount of information in functions of a real variable). The recurrence above is true for $n \geq 3$, boundary cases $p_1 = 0, p_2 = 1/2$. Define

$$G(s) = \sum_{n=1}^{\infty} P_n s^n = \sum_{n=2}^{\infty} P_n s^n = \frac{1}{2}s^2 + \sum_{n=3}^{\infty} P_n s^n$$

$$G(0) = 0$$

Now let's take the sum from $n = 3$ to ∞.

$$\underbrace{\sum_{n=3}^{\infty} nP_n s^{n-1}}_{\frac{d}{ds}[G(s)-1/2\cdot s^2]} = \underbrace{\sum_{n=3}^{\infty} P_{n-1} s^{n-1}}_{s\frac{d}{ds}G(s)} + \underbrace{\sum_{n=3}^{\infty} P_{n-2} s^{n-1}}_{sG(s)}$$

$$G'(s) - s = sG'(s) + sG(s)$$

And now we have a first order differential equation. Separation of variables!

$$\frac{G'(s)}{1 + G(s)} = \frac{s}{1 - s}$$

$$\frac{G'(s)}{1+G(s)} = \frac{1}{1-s} - 1$$

Integrate both sides:

$$\log(1+G(s)) = -s + -\log(1-s) + C$$

Now, if $s = 0, G(s) = 0$, so plug it in to see that $C = 0$. Solving for $G(s)$, we get

$$G(s) = \frac{e^{-s}}{1-s} - 1$$

Now we just need to figure out the coefficients of the power series representing $G(s)$.

$$G(s) = \left(1 - s + \frac{1}{2!}s^2 - \frac{1}{3!}s^3 + \cdots\right)(1 + s + s^2 + \cdots) - 1$$

$$G(s) = (1-1) + (1-1)s + \left(1 - 1 + \frac{1}{2!}\right)s^2 + \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!}\right)s^3 + \cdots$$

How about an even better way to solve this problem! Waring's solution gives a generalization of this.

Waring's Theorem approach

Assume we have n letters we want to stuff. Let X be the number of letters stuffed correctly. We want to find a generating function for X. That is,

$$G_X(s) = \sum_{i=0}^{\infty} \mathbb{P}(X = i)s^i$$

Now define the following:

$$S_m = \sum_{i_1 < \ldots < i_m} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m})$$

which is the sum of probabilities for all subsets of m events. A_{i_j} is an indicator function for i_j-th letter to be stuffed correctly. $S_0 = 1, S_1 = n \cdot \frac{1}{n} = 1, S_2 = \binom{n}{2}\frac{1}{n} \cdot \frac{1}{n-1} = \frac{1}{2!}$. In general,

$$S_m = \binom{n}{m}\frac{1}{n(n-1)\cdots(n-m+1)} = \frac{1}{m!}$$

$$S_m = \mathbb{P}(X = m)\binom{m}{m} + \mathbb{P}(X = m+1)\binom{m+1}{m} + \mathbb{P}(X = m+2)\binom{m+2}{m} + \cdots$$

This is a calculation by conditioning on the total number of letters which are stuffed correctly.

$$S_m = \sum_{i=m}^{n} \binom{i}{m} \mathbb{P}(X = i) = \mathbb{E}\left[\binom{X}{m}\right]$$

Now, get a generating function for S.

$$G_S(s) = \sum_{m=0}^{\infty} \underbrace{\sum_{i=m}^{n} \binom{i}{m} \mathbb{P}(X = i)}_{S_m} s^m = \sum_{i=0}^{n} \sum_{0}^{i} \mathbb{P}(X = i) \binom{i}{m} s^m$$
$$= \sum_{i=0}^{n} \mathbb{P}(X = i)(1 + s)^i = G_X(1 + s)$$

Now we are essentially done. We have figured out that the generating function for S is intimately related to the generating function for X.

$$G_S(s) = G_X(1 + s)$$

$$G_X(s) = G_S(s - 1) = S_0 + S_1(s - 1) + S_2(s - 1)^2 + S_3(s - 1)^3 \cdots$$
$$= \underbrace{(S_0 - S_1 + S_2 - S_3 + \cdots)}_{\mathbb{P}(X=0)} + s\underbrace{(S_1 - 2S_2 + 3S_3 - \cdots)}_{\mathbb{P}(X=1)} + \cdots$$

Thus, the general solution:

$$P(X = i) = \sum_{j=1}^{n} (-1)^{j-1} \binom{j}{i} S_j$$

Problem of the Points

Random walks