Math 191 Notes, 2003 December 4

Generating Functions

Special case: Random variable X with non-negative integer values.

$$\mathbb{P}(X = i) = a_i \qquad G_a(s) = \mathbb{E}(s^X) = \sum_{i=0}^{\infty} a_i s^i$$

Series converges for $s = 1$, hence it converges for $[-1, 1]$. Integrate or differentiate on this entire interval.

$$G_a(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots$$

$$G_a(0) = a_0, \ldots, G_a^{(n)}(0) = n! a_n \qquad \text{(reconstruction)}$$

Let X be called a_i, and Y be b_i, and independent. Let's look at the following:

$$\begin{aligned} G_a(s)G_b(s) &= \sum_{i=0}^{\infty} a_i s^i \sum_{j=0}^{\infty} b_j s^j \\ &= \sum_{n=0}^{\infty} \left(\underbrace{a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0}_{\text{convolution!}} \right) s^n \end{aligned}$$

Thus, multiplication of the generating functions corresponds to convolution of the mass function, which in turn corresponds to the sum of independent random variables. This is why generating functions are so useful.

3 Examples of Generating Functions

1. $X = \{1, 2\}, Y = \{1, 2, 3\}$ with uniform probability. What is the mass function for $Z = X + Y$? Let's first compute the answer directly.

Z	
2	$1/2 \cdot 1/3 = 1/6$
3	$1/2 \cdot 1/3 + 1/2 \cdot 1/3 = 1/3$
4	$1/2 \cdot 1/3 + 1/2 \cdot 1/3 = 1/3$
5	$1/2 \cdot 1/3 = 1/6$

Now let's do it with generating functions to see that we're doing the same thing.

$$
\begin{aligned}
G_X(s) &= \frac{1}{2}s + \frac{1}{2}s^2 \\
G_Y(s) &= \frac{1}{3}(s + s^2 + s^3) \\
G_Z(s) &= G_X(s)G_Y(s) = \frac{1}{6}(s + s^2)(s + s^2 + s^3) \\
&= \frac{1}{6}(s^2 + 2s^3 + 2s^4 + s^5)
\end{aligned}
$$

This is unimpressive[1].

2. Devise two loaded dice such that you get the same probability distribution function for the values 2 through 12 as if you had regular die. For unloaded single die, we have

$$
G_X(s) = \frac{1}{6}(s + s^2 + \cdots + s^6) = \frac{1}{6}s(1 + s + s^2)(1 + s^3)
$$

For two ordinary unloaded dice,

$$
\begin{aligned}
G_{X+Y}(s) &= \frac{1}{36}s(1 + s + s^2)^2 s(1 + s^3)^2 \\
&= \underbrace{\left[\frac{1}{9}s(1 + 2s + 3s^2 + 2s^3 + s^4)\right]}_{G_U(s)} \underbrace{\left[\frac{1}{4}(s + 2s^4 + s^7)\right]}_{G_V(s)}
\end{aligned}
$$

$$
V \quad \begin{cases} 1-1 & p = \frac{1}{8} \text{ each} \\ 4-4 & p = \frac{1}{4} \text{ each} \\ 7-7 & p = \frac{1}{8} \text{ each} \end{cases}
$$

$$
U \quad \begin{cases} 3-3 & p = \frac{1}{6} \text{ each} \\ 1-5 & p = \frac{1}{9} \text{ each} \\ 2-4 & p = \frac{2}{9} \text{ each} \end{cases}
$$

You can check the probabilities work out correctly.

[1] By the way, we could extend to the continuous case by doing $\mathbb{E}(s^X) = \mathbb{E}(e^{tX}) = \mathbb{E}(e^{itX})$ where we get either Laplace or Fourier transforms

3. What is the probability of getting a total of 9 when rolling 3 dice? One die:

$$\begin{aligned} G_X(s) &= \frac{s}{6}(1 + s + \cdots + s^5) \\ &= \frac{s}{6}\frac{1 - s^6}{1 - s} \end{aligned}$$

Three dice

$$\begin{aligned} G_{X+X+X}(s) &= \frac{s^3}{216}(1 - s^6)^3(1 - s)^{-3} \\ &= \frac{s^3}{216}(1 - 3s^6 + 3s^{12} - s^{18})\left(1 + \binom{3}{2}s + \cdots + \binom{n+2}{2}s^n + \cdots\right) \end{aligned}$$

Extract coefficient of s^9, we get

$$\frac{1}{216}\left(-3 + \binom{8}{2}\right) = \frac{25}{216}$$

Almost all of the distributions that we will look at will have easy generating functions.

Common Generating Functions

Bernoulli

$$G(s) = q + ps$$

Sum of n Bernoulli: $(q + ps)^n$

Geometric

$$f_k = pq^{k-1} \qquad G(s) = \sum_{k=1}^{\infty} pq^{k-1}s^k = ps\sum_{k=1}^{\infty}(qs)^{k-1} = \frac{ps}{1 - qs}$$

Poisson

$$f_k = e^{-\lambda}\frac{\lambda^k}{k!} \qquad G(s) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!} = e^{\lambda(s-1)}$$

So if X is Poisson with parameter λ_1, and Y is Poisson with parameter λ_2, then $G_{X+Y}(s) = e^{(\lambda_1+\lambda_2)(s-1)}$ so then we get another Poisson with parameter $\lambda_1 + \lambda_2$!

Moments of a distribution

$$\mathbb{E}(X^n)$$

The first is the mean, the second lets us compute the variance. Let's see how to do this using generating functions.

$$G(s) = \sum_{i=0}^{\infty} f_i s^i, \qquad \text{s.t. } G(1) = 1$$

$$G'(s) = \sum_{i=0}^{\infty} i f_i s^{i-1} \qquad \text{so } G'(1) = \mathbb{E}(X)$$

$$G''(s) = \sum_{i=0}^{\infty} i(i-1) f_i s^{i-2} \qquad \text{so } G''(1) = \mathbb{E}(X^2) - \mathbb{E}(X)$$

Now, slightly better is to set $s = e^t$. We get what's called the **Moment Generating Function** $M(t) = G(e^t)$.

$$M(t) = \sum_{k=0}^{\infty} f_k e^{kt} \qquad M(0) = 1$$

$$M'(t) = \sum_{k=0}^{\infty} k f_k e^{kt} \qquad M'(0) = \mathbb{E}(X)$$

$$M''(t) = \sum_{k=0}^{\infty} k^2 f_k e^{kt} \qquad M''(0) = \mathbb{E}(X^2)$$

Awesome!!

No such thing as a free lunch

Hehe, pretty good but also be careful when you take derivatives. They are a bit complicated! Let's look at the case of a Poisson distribution (the easiest case).

$$M(t) = G(e^t) = e^{\lambda(e^t)-1}$$

$$M'(t) = \lambda e^t e^{\lambda(e^t-1)} \qquad M'(0) = \lambda$$

$$M''(t) = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)} \qquad M''(0) = \lambda^2 + \lambda$$

Check: variance is also $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \lambda^2 + \lambda - (\lambda)^2 = \lambda$.

Problem of the points for the n-th time!

Probability of getting 3 balls before 4 strikes. Next time, we will get a "closed form" solution (haha), in terms of a generating function.

Matching revisited

Strategy here is to get a recurrence relation. Then turn this into a differential equation. Need to calculate p_n, the probability of stuffing all of n letters into the wrong envelope. Steps:

1. Stuff letter $R \neq 1$ into envelope 1. $p = 1 - \frac{1}{n}$.
2. Um...

$$p_n = \left(1 - \frac{1}{n}\right) p_{n-1} + \left(1 - \frac{1}{n}\right) \left(\frac{1}{n-1}\right) p_{n-2}$$

$$p_n = \left(1 - \frac{1}{n}\right) p_{n-1} + \frac{1}{n} p_{n-2}$$