Math 191 Notes, 2003 November 13

Memorize the following

$$N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is properly normalized.

$$\mathbb{E}(X) = \mu$$

$$\text{var}(X) = \sigma^2$$

2 Random Variables

Say we have a double lottery, two random variables X and Y, such that the payoff is the sum of X and Y. If someone tells you that they have something whose payoffs are nearly equal by ratio, fine. If it is by difference, then it is more likely that the payoffs are small.

2 Random variables, X, Y.

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} ds \int_{-\infty}^{y} dt f_{X,Y}(s, t)$$

This is the only definition of the distribution function. Now, let's take some partial derivatives:

$$\frac{\partial}{\partial x} F_{X,Y}(x, y) = \int_{-\infty}^{y} f_{X,Y}(x, t) dt$$

$$\frac{\partial}{\partial y} \frac{\partial}{\partial x} F_{X,Y}(x, y) = f_{X,Y}(x, y)$$

X, Y are independent if the distribution function (and thus) the density function factors as follows:

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} (F_X(x) F_Y(y)) = f_Y(y) f_X(x)$$

Unconscious statistician

$$\mathbb{E}(g(X, Y)) = \int_{-\infty} \infty \int_{-\infty} \infty g(x, y) f(x, y) dx dy$$

Bivariate normal distribution

Density function:

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-p^2)}}$$

Assume $\rho > 0$ and close to 1 ($\rho = 0.8$). Calculate the covariance by completing the square.

$$x^2 - 2\rho xy + \rho^2 y^2 + y^2 - \rho^2 y^2$$

Now the density function factors:

$$f(x,y) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

To calculate the covariance, we calculate

$$\begin{aligned}\mathbb{E}(xy) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy y e^{-y^2/2} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho y^2 e^{-y^2/2} dy = \rho\end{aligned}$$

Now let's find the variance:

$$\begin{aligned}\text{var}(Y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy y^2 e^{-y^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy = 1\end{aligned}$$

Conditional blah-blah

Only condition on events with $\mathbb{P} > 0$ (part of our definition of conditional probability).

$$\begin{aligned}\mathbb{P}(Y \le y | x \le X \le x + dx) &= \frac{\mathbb{P}(Y \le y, x \le X \le x + dx)}{\mathbb{P}(x \le X \le x + dx)} \\ &= \frac{\int_{-\infty}^{y} f(x,v) dv (dx)}{f_X(x)(dx)} = \int_{-\infty}^{y} \frac{f(x,v)}{f_X(x) dv}\end{aligned}$$

is the conditional distribution function. We get the conventional density function by differentiating, that is:

$$f_{Y|X}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Example from the book with diagram.

$$f_{X,Y}(x,y) = \frac{1}{x} \qquad 0 \leq y \leq x < 1$$

we want to know the "marginal" density[1]: $f_X(x) = \int_0^x dy \frac{1}{x} = 1$.

What is $f_{Y|X}$?

$$f_{Y|X}(y,x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{x}$$

Now, create a circular arc intersecting the x-axis at 1. What is the probability $\mathbb{P}(X^2 + Y^2 \leq 1)$? First, define $a(x) = \min(x, sqrt1 - x^2)$ (this would be able to give us a way evaluate the function. Then $A(x) = [0, a(x)]$. To compute this integral use polar coordinates.

$$\iint_{X^2+Y^2 \leq 1} \frac{1}{r\cos\theta} r dr d\theta = \int_0^1 dr \int_0^{\pi/4} \sec\theta d\theta = \log|\sec\theta + \tan\theta|_0^{\pi/4} = \log(1+\sqrt{2})$$

New question:

$$\mathbb{P}(X^2+Y^2 \leq 1 | X = x) = \begin{cases} 1 & \text{if } x < 1/\sqrt{2} \\ \frac{\sqrt{1-x^2}}{x} & \text{if } 1/\sqrt{2} \leq x \leq 1 \end{cases}$$

by inspection of areas. Check:

$$\mathbb{P}(X^2+Y^2 \leq 1) = \int_0^1 dx f_X(x) \int_0^{a(x)} f_{Y|X}(y,x)$$

$$= \int_0^{1/\sqrt{2}} dx + \int_{1/\sqrt{2}}^1 \frac{\sqrt{1-x^2}}{x} dx$$

If we now set $x = \cos\theta$, then we will give something nice (geometrically).

Change of Random Variables

Say we have X, Y. Define: $U = u(X,Y), V = v(X,Y)$ it could be anything, sum, difference, ratio of sine-squared, etc. We have an underlying space of events, and random variables are functions from the space of events to real numbers. So U, V gives you a pair of numbers just as surely as X, Y gives you a pair of numbers.

Assume that $X = x(U,V), Y = y(U,V)$. By and large these are the funcitons that we will be working with. Draw the diagram such that U, V are polar coordinates.

[1] Doesn't seem to match any other use of "margin" or "marginal" that Paul seems to have heard of.