Math 191 Notes, 2003 October 9

Expectation undefined or infinite

Payoff: 1, -3, +5, -7, +9, ...
$f(x) : C, C/9, C/25, C/49, C/81$
How much will you pay me to play this game? Let's calculate $C = \frac{8}{\pi^2}$.[1] But then the expected payoff is $C(1 - \frac{1}{3} + \frac{1}{5} - \cdots) = \frac{8}{\pi^2}\frac{\pi}{4} = \frac{2}{\pi}$. Let's try calculating the variance:

$$\mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 = C + C + C + \cdots$$

So it's extremely risky.

St. Petersburg paradox

First head on turn: 1, 2, 3, 4, 5, ...
You win: 2, 4, 8, 16, 32, ...
Probability: 1/2, 1/4, 1/8, ...
Expectation is infinite!

Another example

Draw dime: I pay you \$1 game ends
Penny: I pay \$1, add another penny, game continues
Let X be the turn on which the dime appears. The probability $\mathbb{P}(X \le x) =$. $f(x) = \frac{1}{x(x+1)}$.

Island casino stock

For any n the sequence of stock prices on days $1, \ldots, n$ is equally likely to be any permutation of the sequence of values $x_1 < x_2 < x_3 < \cdots < x_n$. Expectation, infinite. But there is still an argument for not doing it because it's a very slowly diverging series. There is more than just the numerical expectation to the willingness to play a game.

Matching problem

Envelopes, at random, what's the probability is that nothing is in its right place? $\frac{1}{e}$.

Straightforward probability can be calculated with expectations and indicator functions. Let

$$I_k \begin{cases} 1 & \text{if } k \text{ is good} \\ 0 & \text{if } k \text{ is bad} \end{cases}$$

[1] Euler, *Introductio in Analysin Infinitorum*

Let X be the number of good letters. Let's say that there are n letters total. The probability for some set of r letters all being correct is $\frac{(n-r)!}{n!}$.

$$\mathbb{P}(X=0) = \mathbb{E}[(1-I_1)(1-I_2)\cdots(1-I_n)]$$

But now we just multiply this out, which is a "binomial expansion"!

$$\begin{aligned}
\mathbb{P}(X=0) &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(n-i)!}{n!} \\
&= \sum_{i=0}^{n}(-1)^i\frac{n!}{(n-i)!i!}\frac{(n-i)!}{n!} \\
&= \sum_{i=0}^{n}\frac{(-1)^i}{i!}
\end{aligned}$$