MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Assignment #8

Problems to be discussed in section: All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1,3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 4.4, problem 6.
2. Section 4.4 , problem 8.
3. Section 4.5 , problem 4.
4. Section 4.6, problem 7.
5. Section 4.14, problem 26.
6. Section 4.14, problem 35.

Problems to be handed in on Thursday, November 20: Send email if there appear to be bugs in any of these.

1. Suppose that random variables X and Y have a bivariate normal distribution as specified by equation 10 on p. 100 of G&S. Let $U = X+Y$ and $V = X - Y$. Find a joint density function for U and V, and calculate the variance of each and the covariance of U and V.

2. Suppose that X and Y are independent random variables whose density functions are both given by special cases of the gamma distribution, namely
$$f_X(x) = \lambda^2 x e^{-\lambda x}$$
$$f_Y(y) = \frac{\lambda^3}{2} y^2 e^{-\lambda y}$$
. Find the joint density function $f_{U,V}(u, v)$ for the random variables
$$U = X + Y, V = \frac{X}{X+Y}$$
Show that U and V are independent, and identify the marginal density functions $f_U(u)$ and $f_V(v)$

3. Let X, Y be random variables uniformly distributed on (0,1). Let S be defined by $S = [X/Y]$ where [..] denotes the nearest integer function. So, for example, $[\frac{0.2}{0.5}] = 0$, $[\frac{0.8}{0.3}] = 3$.

   (a) By a geometric argument (or otherwise), calculate the probability $\mathbb{P}(a < X/Y < b)$ for a, $b > 1$.

   (b) Calculate the probability that $S = k$ for integers k=0,1,2,3.

   (c) What is the probability that S is even? You can sum the series if you know that 1 - $\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

4. A darts player always throws his dart so that it hits within the circle of radius 1 centered on the bull's eye. Within this circle the probability density is given by $f_{X,Y}(x, y) = C(1 - x^2 - y^2)$.

   (a) What is the value of C?

   (b) Let $R = \sqrt{X^2 + Y^2}$. Calculate by integration in polar coordinates the distribution function and a density function for R.

   (c) Calculate the marginal distribution function $F_X(x)$ and a marginal density function $f_X(x)$. Remember, when you integrate over y, that the limits of integration depend on x.

   (d) What is the conditional density function $f_{Y|X}(y|X = x)$?

   (e) What is the conditional density function $f_{R|X}(r|X = x)$?

5. You possess one nucleus each of two different species of radionuclide. The first has decay constant λ, which means that the probability that it will survive for time t is $e^{-\lambda t}$. (Note: this is 1 minus the distribution function, but the density function is also exponential.) The second has decay constant 2λ. Let random variable T_1 be the time after which the nucleus of species 1 decays; let random variable T_2 be the time after which the nucleus of species 2 decays.

   (a) Find a density function, the expectation, and the variance for $U = T_1 - T_2$.

   (b) Find a density function and the expectation for T_1, conditioned on $T_1 = T_2$.