MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Assignment #10

Problems to be discussed in section: All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1, 3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 4.13, problem 12. Example 4.13.6 (pp. 136-138) was covered thoroughly, and you can just use the result. There is a (correct) assumption of independence that needs to be justified.
2. Section 4.14, problem 60. This is a lovely problem, but a tedious one. Explain all the steps, but don't take the time to evaluate the integrals.
3. Section 5.12, problem 1. This is just like an example done in lecture, but it takes some work to figure out all the right binomial coefficients!
4. Section 5.12, problem 4. The book's solution makes direct use of the probability mass function. Explain also how you can get the answer from the generating function for the geometric distribution.
5. Section 5.1, problem 2.
6. Section 5.1, problem 9. Invent a simple example for each case.

Problems to be handed in on Thursday, December 11:

1. (a) A helicab driver hopes to pick up passengers along a road that extends from $x = 0$ to $x = 1$. If the driver waits at position $x = s$ and passengers arise at a random location X with uniform density along the road, determine $\mathbb{E}((s - X)^2)$. The result is a function of s. Sketch a graph of it, and then calculate its expectation under the assumption that the position of the helicab driver is also uniformly distributed in [0,1].
   (b) Confirm your calculation by computing an easy integral to determine $\mathbb{E}((X - S)^2)$, where X and S are uniformly and independently distributed in [0,1].
2. Now suppose that the helicab driver of the previous problem hopes to pick up passengers whose position is uniform within a square of side z.
   (a) Using the result of the previous problem, show that if the helicopter is situated at random on the boundary of the square, the expectation of the square of the distance to a passenger is $z^2/2$.

(b) Let $B(z)$ denote the expectation of the square of the distance from the helicopter to the passenger if both have a position that is independently and uniformly distributed within the square. Use Crofton's method to set up a differential equation for $B(z)$, and solve it subject to the boundary condition $B(0) = 0$.

(c) Confirm your solution by a much simpler calculation. You can set up a trivial 4-dimensional integral, or argue from the result of the previous problem, or look at problem 4.14.55 in the text.

3. An unfortunate premedical student must take a semester of calculus in college in order to apply to medical school, and he wants it to be the notorious gut course Math 4. Placement into Math 4 requires a score of precisely 6 out of 15 on the math placement test. There are two options for this placement test. Test A has three questions, on each of which the student has equal probability of getting any score from 0 to 5. Test B has five questions, on each of which the student has equal probability of getting any score from 0 to 3.

   (a) Write down the probability generating functions $G_A(s)$ and $G_B(s)$ for the placement tests, in the same style as problem 5.12.1 in the text.

   (b) By expanding and finding the coefficient of s in each case, determine which leads to the higher probability for a score of precisely 6. If you want to check your answer, it is convenient that both probabilities are calculated by sets.exe, which you can download from the course Web site.

4. Use generating functions to calculate the variance for the hypergeometric distribution. This is example 20 on page 152 of G&S, but the variance calculation was left as an exercise by the authors.

5. An ambitious college graduate decides that before contemplating marriage he should do at least one job that makes him famous, one that makes him rich, and one that makes him happy. Every year he takes a new job. These jobs independently confer one attribute, fame, money, or happiness, each with probability $\frac{1}{3}$. Denote by T_1 the number of jobs that the graduate takes until he acquires one attribute, by T_2 the number of additional jobs that he takes until he acquires a second attribute, and by T_3 the number of additional jobs that he takes until he acquires the third and final attribute. The first of these random variables has a trivial distribution, and the other two each have a geometric distribution. The total time before the student is ready for marriage is $T = T_1 + T_2 + T_3$.

   Write down the probability generating functions for T_1, T_2, T_3, and T. Then write the generating function for T as a sum of partial fractions, and thereby determine the probability mass function for T.

   This is a simplification of problem 5.2.9 in the text, whose solution (to put it mildly) is terse.