Math 191: Solution Set 1

Travis Schedler

September 25, 2001

1. A fair die, when thrown, has an equal chance of falling on any of the numbers 1, 2, 3, 4, 5, or 6. In the case of two dice, the sum of the numbers thrown is between 2 and 12. Both 9 and 10 can be made up in two different ways, 9=3+6=4+5 and 10=4+6=5+5. Is it true that 10 comes up as often as 9 when we throw two dice? Analyze also the case when you throw 3 dice.

Solution to 1. No, the probabilities are not equal, because we must consider separately which number appears on each die. All in all, there are 36 possibilities: die A can have any number 1 through 6, as can die B, and these two are independent events. So, let us consider the event space $\Omega = \{(a,b) : 1 \leq a,b \leq 6\}$ where a denotes the number on die A, and b the number on die b. The subset $X_9 \subset \Omega$ corresponding to a sum of 9 is $X_9 = \{(3,6),(4,5),(5,4),(6,3)\}$ and for 10, it is $X_10 = \{(4,6),(5,5),(6,4)\}$. So the probability of rolling a 9 is $\frac{\#X_9}{\#\Omega} = \frac{4}{36} = \frac{1}{9}$, and the probability of rolling a 10 is $\frac{\#X_10}{\#\Omega} = \frac{3}{36} = \frac{1}{12}$. So it is more likely to roll a 9.

In the case of three dice, considering event space $\Omega = \{(a,b,c) : 1 \leq a,b,c \leq 6$, with $\#\Omega = 6^3 = 216$, we have $X_9 = \{(1,2,6),(1,3,5),(1,4,4),(2,2,5),(2,3,4),(3,3,3)\}$ together with permutations of these 6 ordered pairs. There are 3 ordered pairs which consist of all a,b,c distinct, which have 6 permutations each; two ordered pairs where a,b,c include exactly two distinct values, which have 3 permutations each; and one ordered pair where $a = b = c$, which has only one permutation. So $\#X_9 = 6+6+3+3+6+1 = 25$, and $\mathbb{P}(X_9) = \frac{25}{216}$. On the other hand, $X_10 = \{(1,3,6),(1,4,5),(2,2,6),(2,3,5),(2,4,4),(3,3,4)\}$ and permutations. So $\#X_10 = 6+6+3+6+3+3 = 27$, and $\mathbb{P}(X_10) = \frac{27}{216} = \frac{1}{8}$. So the chance is better of getting a 10 than a 9.

Remark. The most likely sum, rolling any number of dice more than 1, is the middle possible value; in the case of 2 dice it is $\frac{2+12}{2} = 7$, with a probability of $\frac{1}{6}$; in the case of 3 dice it is $\frac{3+18}{2} = 10.5$ so actually 10 and 11 have the same maximum probability of $\frac{1}{8}$. The probability distribution, it is easy to see, is symmetric about the center maximum and the probability monotonically increases approaching the center.

2. A paradox. In four throws of a single die the probability that we get at least one ace [=1] is more than 1/2 [Please find this probability!!]. Therefore the probability of getting a double ace [that is two aces in a row] at least once is 6 times less. So the probability of getting a double ace at least once in 24 throws should also be more than 1/2. On the other hand, direct computations show that this probability is actually less than 1/2. [Please find this probability!!]. How would you explain the "paradox"?

Remark. The problem 2 is tricky.

Solution of 2. The probability in four throws of getting no aces is $(\frac{5}{6})^4 = \frac{625}{1296}$, so the probability of getting at least one ace is $\frac{671}{1296}$, which is larger than 1/2. However, there is no reason why the probability of getting a double ace should be 6 times less! In fact, the probability of getting a double ace can be computed as follows: For two rolls, the probability is clearly $\frac{1}{36}$. Let $\mathbb{P}(n)$ be the probability of obtaining a double ace in n rolls. Well, given $n+1$ rolls, we can divide into the case where the first die reads 1 or not. If it reads 1, then the probability is 1/6 that we will obtain a double ace with the next die, and $5/6\mathbb{P}(n-1)$ we will obtain a double ace later on. If the first die does not read one, then the probability is $\mathbb{P}(n)$ that we will obtain a double ace later. Thus, we have the inductive formula $\mathbb{P}(n+1) = \frac{1}{6}(\frac{1}{6} + \frac{5}{6}\mathbb{P}(n-1)) + \frac{5}{6}\mathbb{P}(n)$. Also, $\mathbb{P}(1) = 0$ and $\mathbb{P}(2) = \frac{1}{36}$. Iterating this, we get $\mathbb{P}(3) = \frac{11}{216}, \mathbb{P}(4) = \frac{2}{27}, \ldots$, and $\mathbb{P}(24) \sim .434$.

The explanation of the paradox is that we cannot simply divide the probability of at least one ace by 6 to get the probability of a double ace in the case of four dice; in fact, we see that the probability, 2/27, of a double ace is less than a sixth of the probability, $\frac{671}{1296}$, of getting at least one ace. This is the main error, although there is a less egregious one: the probability of getting a double ace in 24 rolls, .434, is slightly less (1%) than six times the probability of obtaining a double ace in four rolls ($4/9 = .444$), not exactly six as the problem claims.

3. Peter and Robert are playing a fair game [that is, both have the same chances of winning], and have agreed that whoever wins 6 rounds first gets the whole prize. They had to stop when Peter won 5 and Robert 3 rounds. How should the prize be divided fairly?

Solution to 3. We should compute the probability that Peter would win if the game continued, and also the probability that Robert would win, and give each person the portion of the prize corresponding to the person's probability. It is easier to compute Robert's probability: he has to win 3 games in a row, so the probability is $(1/2)^3 = 1/8$. Peter's probability, therefore, must be 7/8 of winning should the games continue. Hence, Peter should be awarded 7/8 of the prize and Robert 1/8.

Section 1.2, #1. Let $\{A_i : i \in I\}$ be a collection of sets. Prove 'De Morgan's Laws':

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c, \quad \left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c.$$

Solution to Section 1.2, #1. We see that $x \in \left(\bigcup_i A_i\right)^c$, is the same as $x \notin \left(\bigcup_i A_i\right)$, which is the same as $x \notin A_i$ for any i, which is the same as $x \in A_i^c$ for all i, which is the same as $x \in \bigcap_i A_i^c$. Similarly, $x \in \left(\bigcap_i A_i\right)^c$ is the same as $x \notin \bigcap_i A_i$, i.e. $x \notin A_i$ for at least one i, i.e. $x \in A_i^c$ for at least one i, i.e. $x \in \bigcup_i A_i^c$. Note that all of the statements, "is the same as", or "i.e." are biconditionals (the implications go both directions.)

Section 1.2, #2. Let A and B belong to some σ-field $\mathcal{F}$. Show that $\mathcal{F}$ contains the sets $A \cap B$, $A \setminus B$, and $A \triangle B$.

Solution to Section 1.2, #2. We know that $(A^c \cup B^c)^c \in \mathcal{F}$, but by DeMorgan's law and the law $(X^c)^c = X$, this implies just that $A \cap B \in \mathcal{F}$. Similarly, $A \setminus B = A \cap (B^c)$ and $A \triangle B = (A \setminus B) \cup (B \setminus A)$ sequentially prove that $A \setminus B, A \triangle B \in \mathcal{F}$.

Section 1.2, #3. A conventional knock-out tournament (such as that at Wimbledon) begins with 2^n contestants and has n rounds. There are no play-offs for the positions $2, 3, \ldots, 2^n - 1$, and the initial table of draws is specified. Give a concise description of the sample space of all possible outcomes.

Solution to Section 1.2, #3. One description is based on ordering the games to be played, of which there are $2^n - 1$, and ordering the contestants, 1 through 2^n, so that in round m, the two undefeated players numbered in each of the ranges $1 + 2^m(k-1) \ldots 2^m(k)$, as k ranges from 1 to 2^{n-m}, play against each other in game $(2^m - 1) + k$. Then, we assign to each of the $2^n - 1$ games a 1 if the player with a lower number wins, and a 0 otherwise. This describes completely the possible outcomes, which has size $2^{2^n - 1}$.

Section 1.2, #5. Which of the following are identically true? For those that are not, say when they are true.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
(b) $A \cap (B \cap C) = (A \cap B) \cap C$;
(c) $(A \cup B) \cap C = A \cup (B \cap C)$;
(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Solution to Section 1.2, #5. (a) true; (b) true; (c) true exactly when $A \subset C$; (d) true.

Section 1.3, #1. Let A and B be events with probabilities $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$. Show that $\frac{1}{12} \le \mathbb{P}(A \cap B) \le \frac{1}{3}$, and give examples to show that both extremes are possible. Find corresponding bounds for $\mathbb{P}(A \cup B)$.

Solution to Section 1.3, #1. We know that

$$\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A \cup B) \le 1, \tag{1}$$

so $\frac{1}{12} \le \mathbb{P}(A \cup B)$. On the other hand, $\mathbb{P}(A \cap B) \le \mathbb{P}(B) = \frac{1}{3}$, proving the inequalities. As an example when both extremes are attained, first consider the case where in a game with outcomes $1 \ldots 12$, each of which is equally likely, $A = \{1, \ldots, 9\}$ and $B = \{9, \ldots, 12\}$. In this case, $\mathbb{P}(A \cap B) = \frac{1}{12}$. Now, consider the same game but with $A = \{1, \ldots, 9\}$ and $B = \{1, \ldots, 4\}$. Now $\mathbb{P}(A \cap B) = \frac{1}{3}$. The bounds for $\mathbb{P}(A \cup B)$ are just $\frac{3}{4} \le \mathbb{P}(A \cup B) \le 1$, which can be obtained from the bounds we have by using (1), and extremal examples are the same as those given.

Section 1.3, #2. A fair coin is tossed repeatedly. Show that, with probability one, a head turns up sooner or later. Show similarly that any given finite sequence of heads and tails occurs eventually with probability one. Explain the connection with Murphy's Law.

Solution to Section 1.3, #2. Given n flips, the probability that no head appears is $\frac{1}{2^n}$. Thus, the probability that no head will ever appear in infinitely many flips is $\le \frac{1}{2^n}$. This is true for every n, so the probability is zero in infinitely many flips that there will never be a head. Hence, the probability is one that a head will eventually occur. Similarly, given any finite sequence of length m of heads and tails, we may consider the probability that in km flips, the sequence does not appear in any of the k ranges $m(j-1)+1 \ldots mj$ for $1 \le j \le k$. This probability is $\left(\frac{2^m - 1}{2^m}\right)^k$. Also, for any k it is clear that the probability that the sequence will never appear in an infinite number of flips is less than or equal to this. On the other hand, since $\frac{2^m-1}{2^m} < 1$, the sequence $\left(\frac{2^m-1}{2^m}\right)^k$ converges to 0 as $k \to \infty$. Hence, the probability is zero that the sequence will never appear; hence one that the sequence will eventually appear. The connection with Murphy's law is that anything (bad), however improbable, will eventually occur (with probability one).

Section 1.3, #3. Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two with stars on. If the cups are placed randomly onto the saucers (one each), find the probability that no cup is upon a saucer of the same pattern.

Solution to Section 1.3, #3. Label the cups $1, 2, 3, 4, 5, 6$, and the saucers also $1, 2, 3, 4, 5, 6$, so that the first two are red, the next two white, the next two have stars on. Let us assign each cup to a saucer. The number of ways of doing this with no restrictions is $6! = 720$. Now let us count the number of assignments so that no cup has a matching saucer. First suppose that all cups of the same pattern are assigned to a saucer of the same pattern. If red is assigned to white, then white must be assigned to star and star to red; here there are 8 possibilities. Similarly, red $\to$ star $\to$ white $\to$ red has 8 possibilities. Now consider the case where two cups of the same pattern get assigned to two differently-patterned saucers. If red cups are assigned to both white and star saucers, then clearly white cups cannot both be assigned to star saucers; if they are both assigned to red saucers than one star cup will be assigned to a star saucer, which isn't allowed. Hence, every pattern of cup is assigned to the two different patterns available to it. The number of ways to do this is just $4 * 2 = 8$ for the assignments available to red, and given these assignments, one white cup will be assigned to the remaining star cup and the other to any red cup, for a total of 4 more possibilities. Finally, we can choose which star cup goes to a red and which to a white. So there are a total of $4 * 2 * 4 * 2 = 64$ ways to do this. In total there are $8 + 8 + 64 = 80$ ways of assigning cups to saucers which obeys the restrictions; thus the probability is $80/720 = 1/9$ that the assignment will randomly obey this restriction.

Section 1.3, #4. Let $A_1, A_2, \ldots, A_n$ be events where $n \geq 2$, and prove that

$$\mathbb{P}\left(\cup_{i=1}^n A_i\right) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\mathbb{P}(a_1 \cap A_2 \cap \cdots \cap A_n). \quad (2)$$

In each packet of Corn Flakes may be found a plastic bust of one of the last five Vice-Chancellors of Cambridge University, the probability that any given packet contains any specific Vice-Chancellor being $\frac{1}{5}$, independently of all other packets. Show that the probability that each of the last three Vice-Chancellors is obtained in a bulk purchase of six packets is $1 - 3(\frac{4}{5})^6 + 3(\frac{3}{5})^6 - (\frac{2}{5})^6$.

Solution to Section 1.3, #4. The proof of (2) is by induction. Clearly the statement holds for $n = 1$, and for $n = 2$ it holds by law (1). Now suppose that (2) is true for n. To prove it for $n+1$, we find that

$$\mathbb{P}\left(\cup_{i=1}^{n+1} A_i\right) = \mathbb{P}\left(\cup_{i=1}^{n} A_i\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\left(\cup_{i=1}^{n} A_i\right) \cap A_{n+1}\right)$$
$$= \mathbb{P}(\left(\cup_{i=1}^{n} A_i\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\cup_{i=1}^{n} A_i \cap A_{n+1}\right). \quad (3)$$

Now if we substitute in (2) for the two matching expressions, we get exactly the RHS of (2) for $n+1$.

Now, the probability that the last three Vice-Chancellors are contained in any six packets is given as follows. Let A_1, A_2, A_3 respectively be the events where the last Vice-Chancellor is not contained in any of the packets, the second-to-last isn't, and the third-to-last isn't. Then $\mathbb{P}(A_i) = (\frac{4}{5})^6$, $\mathbb{P}(A_i \cap A_j)(\frac{3}{5})^6$, and $\mathbb{P}(A_1 \cap A_2 \cap A_3) = (\frac{2}{5})^6$, where i, j are any distinct integers from 1 to 3. Hence, using the formula, we obtain $\mathbb{P}(A_1 \cup A_2 \cup A_3) = 3(\frac{4}{5})^6 - 3(\frac{3}{5})^6 + (\frac{2}{5})^6$. This is the probability that none of the last three Vice-Chancellors are in the six packets; subtracting it from one gives the desired probability and proves the desired formula.