MATH 191 - PROBLEM SET 1
SOLUTIONS

p. 25, # 19 Nobody seemed to have trouble with this one.

p. 52, #5 Let $X = \{(d_1, \ldots, d_6) \mid d_i \in \{1, \ldots 6\}\}$. The cardinality of this set is 6^6. Let A be the subset of X consisting of all sequences with at least one 1. The complement of A in X is the set of all sequences with no $1's$, $\{(d_1, \ldots, d_6) \mid d_i \in \{2, \ldots 6\}\}$. $N(X - A) = 5^6$. A and $X - A$ are disjoint, $P(X - A) = \left(\frac{5}{6}\right)^6$, and $P(A) = 1 - \frac{5^6}{6^6} \sim .665$.

Now let Y be the set of all 12−tuples of numbers between 1 and 6. $N(Y) = 6^{12}$. If we let B be the set of all tuples with at least 2 occurances of the number 1, then the compliment of B in Y is

$$\{(d_1, \ldots, d_{12}) \mid d_i \neq 1 \forall i\} \coprod \{(d_1, \ldots, d_{12} \mid \exists i, d_i = 1, d_j \neq d_i \forall j \neq i)\}.$$

There are $\binom{12}{1}$ different ways to place the single 1, and the remaining spaces may be filled by any number different from 1. Therefore, there are $\binom{12}{1}5^{11} = (12)5^{11}$ ways of doing this.Therefore,

$$P(B) = 1 - P(Y - B) = 1 - \left(\frac{5^{12}}{6^{12}} + \frac{12 \cdot 5^{11}}{6^{12}}\right) = 1 - \frac{17 \cdot 5^{11}}{6^{12}} \sim .618.$$

#11. Let's assume that the man get's the right key on the r−th try. This means that he has chosen r distinct keys from a collection of n, without replacement and with order mattering, but that the last one is the one that fits the door. Essentially, this fixes the last key as the appropriate one, and we have just chosen $r-1$ keys from the $n - 1$ keys. The order of the choice matters, and we draw without replacement, so there are $(n - 1)_{r-1}$ distinct ways to do this.

On the other hand, the total number of r strings is the same as the number of ways to draw r keys from a collection of n, with order mattering and without replacement. This is just $(n)_r$. Assuming that each event has the same possibility, the probability that the right key is the r^{th} one chosen is just:

$$P(r) = \frac{(n-1)_{r-1}}{(n)_r} = \frac{\frac{(n-1)!}{(n-r)!}}{\frac{n!}{(n-r)!}} = \frac{1}{n}.$$

#17. The order of A and B doesn't matter, so we can just compute assuming that A is the left most and then multiply by 2.

Since there are n objects to be placed in n slots, with order differentiation, we know that there are $n!$ totally possible arrangements.

There are $n - (r + 2) + 1$ different slots for the $r + 2$ person string from A to B to go in the line up.

This gives us all possible places for A and B, as B is just $r + 2$ places behind A. How do we place the remaining $n - 2$ objects into the $n - 2$ remaining spaces? Since order matters, this is just $(n - 2)!$. Multiplying these together (we want both conditions to be satisfied) tells us that there are $2(n-r-1)(n-2)!$ ways to arrange the people so that the conditions are satisfied. Dividing by the total number of states and assuming equal probability for each state gives the probability as:

$$P(r) = \frac{2(n-r-1)}{n(n-1)}.$$

If we now arrange the people in a circle, then there are a total of $n!$ different ways of doing this. The reason for this is that there is no distinguished initial chair, so we lose a factor of n from the linear ordering (we are actually dividing by the action of the cyclic group of order n on the circle). Additionally, because we can't tell any chairs apart, there is only 1 way to put A and B (following that arc-reduction convention used in the book.) So, now we have $n-2$ elements to place in $n-2$ slots without replacement and with order mattering. There are $(n-2)!$ ways to do this. Assuming that all possible states are equally likely, the probability is therefore

$$P(r) = \frac{(n-2)!}{(n-1)!} = \frac{1}{n-1}.$$

#21. Everybody got this one right.

#28. The same here. **#38.** In the part (a) everybody got it right:

$$P(r_1, \ldots, r_n) = \frac{\prod_{i=1}^{n} \binom{N}{r_i}}{\binom{nN}{r}}.$$

In the part (b) everybody approximated with the correct value of

$\frac{r!}{r_1! \ldots r_n! n^r}$,

but virtually no one used the Stirling's formula, which was the point of the problem.

#41. Everybody got this one right.

p. 62, #24 Everybody got this one as well. Points were taken for insufficient explanation.