# Math 191 Solution Set 4 and 5

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III.6 Figure 1 makes the geometry fairly obvious. Given a path from $(0,0)$ to $(2n+2,0)$ with all interior vertices strictly above the axis, we see that it must reach $(1,1)$ and $(2n+1,1)$ (because otherwise it reaches $(1,-1)$ or $(2n+1,-1)$). Chopping off the first and last segments of the path and translating the entire path down and left by one unit, we obtain a length $2n$ path from $(0,0)$ to $(2n,0)$ with all vertices above or on the axis. Conversely, given a path from $(0,0)$ to $(2n,0)$ with all vertices above or on the axis, we translate it up and right by one unit and add segments from $(0,0)$ to $(1,1)$ and $(2n+1,1)$ to $(2n+2,0)$ to obtain a path from $(0,0)$ to $(2n+2,0)$ with all interior vertices strictly above the axis. These two transformations are clearly inverses and prove that the number of paths of each time is the same.

Hence

$$\begin{aligned} P\{S_1 \geq 0, \ldots, S_{2n-1} \geq 0, S_{2n} = 0\} &= \frac{1}{2^{2n}}(\#\text{ paths of second kind}) \\ &= 2 \cdot \frac{1}{2^{2n+2}}(2 \cdot \#\text{ paths of first kind}) \\ &= 2 \cdot \frac{1}{2^{2n+2}}(\#\text{ length } (2n+2) \text{ paths with first} \\ &\quad \text{return at } 2n+2) \\ &= 2f_{2n+2}. \end{aligned}$$

The third equality is due to the fact that of all length $(2n+2)$ paths with first return at $2n+2$, exactly half of them go above the axis.

III.8 Let $A$ denote the set of length $2n$ paths above the axis except at the origin. Let $B$ denote the set of length $2n-1$ paths above or on the axis. Given a path in $A$, we see that it has to reach $(1,1)$. Chop off the first segment and shift the path down and left by one unit to obtain a path in $B$. Conversely, given a path in $B$, shift it up and right by one unit and add a segment between $(0,0)$ and $(1,1)$ to obtain a path in $A$. These inverse transformations between $A$ and $B$ show that they have the same size. Hence $P\{S_1 > 0, \ldots, S_{2n} > 0\} = \frac{\#A}{2^{2n}} = \frac{1}{2} \cdot \frac{\#B}{2^{2n-1}} = \frac{1}{2} P\{S_1 \geq 0, \ldots, S_{2n-1} \geq 0\}$.

III.9 For $i = 0, \ldots, n-1$, let $v_{2i}$ denote the number of length $2n$ paths with $r$th return to the axis occurs at $(2i, 0)$ and without a return afterwards, and let $w_{2i}$ denote the number of length $2n$ paths with $r$th return to the axis occurs at $(2i, 0)$ and with a return at $(2n, 0)$ as well. Then

$$\begin{aligned} v_{2i} &= \#\{\text{length } 2i \text{ paths with } r\text{th return at } (2i,0)\} \cdot \\ &\quad \#\{\text{length } 2n-2i \text{ paths with no return}\} \\ &= \#\{\text{length } 2i \text{ paths with } r\text{th return at } (2i,0)\} \cdot \\ &\quad \#\{\text{length } 2n-i \text{ paths with return at the end}\} \\ &= w_{2i}. \end{aligned}$$

The second equality is simply equation (3.1). Now the two desired probabilities are $\frac{1}{2^{2n}} \sum_{i=0}^{n-1} v_{2i}$ and $\frac{1}{2^{2n}} \sum_{i=0}^{n-1} w_{2i}$, so the probabilities are equal.

III.10

$$\begin{aligned} z_{r,2n} &= P\{r \text{ returns up to and including epoch } 2n\} \\ &= P\{r\text{th return at epoch } 2n\} + P\{r \text{ returns before epoch } 2n\} \\ &= \rho_{r,2n} + P\{\text{return at epoch } 2n \text{ and preceded by at least } r \text{ returns}\} \\ &= \rho_{r,2n} + \sum_{i \geq r+1} P\{i\text{th return at epoch } 2n\} \\ &= \rho_{r,2n} + \rho_{r+1,2n} + \rho_{r+2,2n} + \ldots \\ &= \varphi_{r,2n-r} + \varphi_{r+1,2n-r-1} + \varphi_{r+2,2n-r-2} + \ldots \\ &= \frac{1}{2} \sum_{i \geq 0} (p_{2n-r-i-1,r+i-1} - p_{2n-r-i-1,r+i+1}). \end{aligned}$$

Now $p_{2n-r-i-1,r+i+1}$ is the probability of reaching $(2n-r-i-1, r+i+1)$. The previous step must be $(2n-r-i-2, r+i)$ or $(2n-r-i-2, r+i+2)$. From each of these points, the probability is $1/2$ of reaching $(2n-r-i-1, r+i+1)$. Hence $p_{2n-r-i-1,r+i+1} = \frac{1}{2} p_{2n-r-i-2,r+i} + \frac{1}{2} p_{2n-r-i-2,r+i+2}$, implying that

$$p_{2n-r-i-2,r+i} - p_{2n-r-i-1,r+i+1} = p_{2n-r-i-1,r+i+1} - p_{2n-r-i-2,r+i+2}.$$

Thus

$$
\begin{aligned}
z_{r,2n} &= \frac{1}{2}\sum_{i\geq 0}(p_{2n-r-i-1,r+i-1} - p_{2n-r-i-1,r+i+1}) \\
&= \frac{1}{2}p_{2n-r-1,r-1} + \frac{1}{2}\sum_{i\geq 0}(p_{2n-r-i-2,r+i} - p_{2n-r-i-1,r+i+1}) \\
&= \frac{1}{2}p_{2n-r-1,r-1} + \frac{1}{2}\sum_{i\geq 0}(p_{2n-r-i-1,r+i+1} - p_{2n-r-i-2,r+i+2}) \\
&= \frac{1}{2}p_{2n-r-1,r-1} + \frac{1}{2}p_{2n-r-1,r+1} \\
&= \frac{1}{2^{2n-r}}\binom{2n-r-1}{n-1} + \frac{1}{2^{2n-r}}\binom{2n-r-1}{n} \\
&= \frac{1}{2^{2n-r}}\binom{2n-r}{n}.
\end{aligned}
$$

The second equality involves rearranging the terms. The third equality is substitution by the equality derived above. The fourth equality results from cancellation of terms. (Thanks to Thomas Castillo for finding a way to get the sum to telescope.)

III.12 Let $A_i$ denote the paths from $(0,0)$ to $(2n,0)$ that reaches the line $y=i$. Given a path in $A_i$, if $(t,i)$ is the last time the path is on $y=i$, then reflect the path from $x=t$ to $2n$ through the the line $y=i$. This produces a path from $(0,0)$ to $(2n,2i)$. Conversely, given a path from $(0,0)$ to $(2n,2i)$, if $(t,i)$ is the last time the path is on $y=i$, then reflect the path from $x=t$ to $2n$ through the the line $y=i$. This produces a path in $A_i$. Hence we have a bijection, and $\#A_i = N_{2n,2i}$.

Now a path has maximum $k$ if and only if it reaches $y=k$ but not $y=k+1$. Thus $A_k - A_{k+1}$ is the set of paths with maximum $k$ and $S_{2n}=0$. So the desired probability is $\frac{1}{2^{2n}}(\#A_k - \#A_{k+1}) = \frac{N_{2n,2k}}{2^{2n}} - \frac{N_{2n,2k+2}}{2^{2n}} = P\{S_{2n}=2k\} - P\{S_{2n}=2k+2\}$.

Extra As asserted in class, the basic strategy is to walk past $k$ stones and pick up the first stone thereafter that is larger than the first $k$ stones. The problem is to maximize the probability of winning as a function of $k$.

Let $A_i$ be the event that the largest stone is the $i$th one along the path. If $A_i$ occurs and $i>k$, then the peasant wins if and only if the largest stone among the first $i-1$ is one of the first $k$ stones. Why? If this condition holds, then stones $k+1$ through $i-1$ will be smaller than some stone among the first $k$, so the peasant will pick up the $i$th stone and win. Conversely, if the condition does not hold, then the peasant will pick up the largest stone among the first $i-1$ and lose.

But the probability that the largest stone among the first $i-1$ is one of the first $k$ is just $\frac{k}{i-1}$. Hence

$$\begin{aligned} P(\text{winning}) &= \sum_{i=k+1}^{n} P(\text{winning}|A_i)P(A_i) \\ &= \sum_{i=k+1}^{n} \frac{k}{i-1} \cdot \frac{1}{n} \\ &= \frac{k}{n}\left(\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n-1}\right) \\ &\approx \frac{k}{n}\int_{k}^{n} \frac{1}{x} dx \\ &= \frac{k}{n}(\ln n - \ln k). \end{aligned}$$

The derivative of this with respect to $k$ is $\frac{\ln n}{n} - \frac{1}{n}(\ln k + 1) = \frac{1}{n}\ln\frac{n}{ke}$. This is zero when $k = n/e$. Hence the optimal $k$ is approximately $n/e$, yielding a winning probability of approximately $1/e \approx 0.368$.