MATH 155 PROBLEM SET 5 (DUE TUESDAY NOV. 20) NO EXTENSIONS PERMITTED!

(1) (a) Define a variation of the RSK algorithm that gives a bijection between $0,1$ matrices A of finite support and pairs (P,Q) such that the transpose of P and Q are semistandard tableaux with $\mathrm{sh}(P) = \mathrm{sh}(Q)$, $\mathrm{col}(A) = \mathrm{type}(P)$, and $\mathrm{row}(A) = \mathrm{type}(Q)$.

(b) Prove that

$$\prod_{i,j}(1 + x_i y_j) = \sum_{\lambda} s_{\lambda}(x) s_{\lambda'}(y).$$

(c) Let ω_y denote ω acting on the y variables only. Show that $\omega_y \prod(1 - x_i y_j)^{-1} = \prod(1 + x_i y_j)$.

(d) For every partition λ show that $\omega(s_\lambda) = s_{\lambda'}$, where λ' is the transpose of λ.

(2) (a) The *Durfee size* of a partition λ is the number of diagonal squares (i,i) in its shape. Show that $\chi^\lambda(\alpha) = 0$ if the Durfee size of λ is bigger than the number of parts of α.

(b) Show that if λ is a partition of n and μ is a partition of $k \leq n$, then

$$\chi^\lambda(\mu 1^{n-k}) = \sum_{\nu} f^{\lambda/\nu} \chi^\nu(\mu).$$

Here the sum is over all partitions ν of k, and $\mu 1^{n-k}$ denotes the partition obtained by adding $n-k$ parts of length 1 to μ.

(c) Let $0 \leq s \leq n-1$ and λ be a partition of n. Show that if $w \in S_n$ is an n-cycle, then $\chi^\lambda(w)$ is equal to $(-1)^s$ if $\lambda = (n-s, 1^s)$ and is equal to 0 otherwise.

(3) (*) Recall that the symmetric group S_n is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$ together with the relations $s_i^2 = 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. Define a function f on S_n by defining $f(w)$ to be the smallest number k of simple transpositions in any expression $s_{i_1} s_{i_2} \dots s_{i_k}$ for w. Describe $f(w)$ in another way, in terms of the list expression for w, i.e. the list of numbers $(w(1), w(2), \dots, w(n))$. Then prove that your description is equivalent to the first definition of f.