

MATH 155 PROBLEM SET 4 (DUE THURSDAY NOV. 8)

- (1) (a) Let  $e_k(n) = e_k(x_1, x_2, \dots, x_n)$  and similarly for the homogeneous symmetric functions. Show that for  $n \geq 1$ , we have:

$$\begin{aligned} e_k(n) &= e_k(n-1) + x_n e_{k-1}(n-1), \\ h_k(n) &= h_k(n-1) + x_n h_{k-1}(n), \end{aligned}$$

and  $e_k(0) = h_k(0) = \delta_{k,0}$ , where  $\delta_{k,0}$  is the Kronecker delta.

- (b) The (signless) Stirling numbers of the first kind are defined by  $c(n, k)$  is equal to the number of  $\pi \in S_n$  with  $k$  disjoint cycles. The Stirling numbers of the second kind are defined by  $S(n, k)$  is equal to the number of partitions of  $\{1, \dots, n\}$  into  $k$  subsets. Show that for  $n \geq 1$ , we have:

$$\begin{aligned} c(n, k) &= c(n-1, k-1) + (n-1)c(n-1, k) \\ S(n, k) &= S(n-1, k-1) + kS(n-1, k) \end{aligned}$$

and  $c(0, k) = S(0, k) = \delta_{k,0}$ .

- (c) Show that:

$$\begin{aligned} \binom{n}{k} &= e_k(1^n) = h_k(1^{n-k+1}) \\ c(n, k) &= e_{n-k}(1, 2, \dots, n-1), \\ S(n, k) &= h_{n-k}(1, 2, \dots, k). \end{aligned}$$

- (2) (a) For a nonempty partition  $\lambda$ , prove that the skew Schur function  $s_{\lambda/1}$  equals the sum of Schur functions  $s_\mu$  over all partitions  $\mu$  obtained from  $\lambda$  by removing a corner box. Hint: there is a very short (four-line-ish) proof!
- (b) for a partition  $\lambda$  whose Young diagram has at least two rows and at least two columns, prove that  $s_{\lambda/(2)} - s_{\lambda/(1^2)}$  equals  $\sum s_\mu - \sum s_\nu$  over all partitions  $\mu$  obtained from  $\lambda$  by removing a horizontal domino and all partitions  $\nu$  obtained from  $\lambda$  by removing a vertical domino. Hint: the involution  $\omega$  on symmetric functions might be useful here.
- (c) Find all partitions  $\lambda$  such that  $s_{\lambda/(2)} = s_{\lambda/(1^2)}$ .

- (3) The *Jacobi-Trudi* formula was the original definition of the Schur polynomials:

$$s_\lambda(x_1, \dots, x_m) = \frac{\det((x_j)^{\lambda_i+m-i})_{1 \leq i, j \leq m}}{\det((x_j)^{m-i})_{1 \leq i, j \leq m}}. \quad (1)$$

Let  $t_\lambda$  be the right-hand-side of (1). Prove (1) by showing that

$$h_p(x)t_\lambda = \sum_{\nu} t_\nu, \quad (2)$$

the sum over all  $\nu$ 's that are obtained from  $\lambda$  by adding  $p$  boxes, with no two in a column. For any  $\ell_1 > \ell_2 > \dots > \ell_m \geq 0$ , let  $a(\ell_1, \dots, \ell_m) = |(x_j)^{\ell_i}|$ . Show that (2) is equivalent to

$$a(\ell_1, \dots, \ell_m) \prod_{i=1}^m (1 - x_i)^{-1} = \sum a(n_1, \dots, n_m), \quad (3)$$

the sum over all  $n_1 \geq \ell_1 > n_2 \geq \ell_2 > \dots > n_m \geq \ell_m$ . Prove (3) by induction on  $m$ , expanding the determinant  $a(\ell_1, \dots, \ell_m)$  along the top row.

(4) The Schur functions can also be expressed as follows:

$$s_\lambda(x) = \det(h_{\lambda_i+j-i}(x))_{1 \leq i, j \leq k} \quad (4)$$

For  $p$  between 1 and  $m$ , let  $e_r^{(p)}$  denote the  $r$ th symmetric polynomial in the variables  $x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_m$ . Show that

$$\left( \sum h_i(x) t^i \right) \left( \sum e_r^{(p)}(-t)^r \right) = (1 - x_p t)^{-1}.$$

Deduce the formula  $\sum_{j=1}^m h_{q+j-m}(x) (-1)^{m-j} e_{m-j}^{(p)} = (x_p)^q$ , and hence the matrix identity

$$(h_{\lambda_i+j-i}(x))_{1 \leq i, j \leq m} ((-1)^{m-j} e_{m-j}^{(p)})_{1 \leq j, p \leq m} = ((x_p)^{\lambda_i+m-i})_{1 \leq i, p \leq m}.$$

Deduce (4) from this identity.

(5) Show that

$$s_\lambda(1, x, x^2, \dots, x^{m-1}) = x^r \prod_{i < j} \frac{x^{\lambda_i - \lambda_j + j - i} - 1}{x^{j-i} - 1},$$

where  $r = \lambda_2 + 2\lambda_3 + \dots = \sum (i-1)\lambda_i$ , the product over all  $1 \leq i < j \leq m$ .

(6) Specializing  $x = 1$  in the previous problem, show that

$$s_\lambda(1, \dots, 1) = \prod_{i < j} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

What does this formula enumerate?

(7) (just for fun; not for credit) Define a strict Gelfand-Tsetlin pattern of order  $r$  to be a Gelfand-Tsetlin pattern with first row  $1, 2, \dots, r$  for which every row is *strictly* increasing. Let  $M(r)$  be the number of strict Gelfand-Tsetlin triangles of order  $r$ . Conjecture (and prove!?) a formula for  $M(r)$ .