MATH 155 PROBLEM SET 3 (DUE THURSDAY OCT. 25)

(1) Let $V := \{(z_1, \ldots, z_n) \mid z_1 + \cdots + z_n = 0\} \cong \mathbb{C}^{n-1}$. The symmetric group S_n acts on V by permutations of the coordinates.
   (a) Find the Gelfand-Tsetlin basis of the representation V. Hint: find the basis $v_1, \ldots, v_{n-1}$ of V such that each v_i is a common eigenvector of the Jucys-Murphy elements $X_i = (1i) + (2i) + \cdots + (i-1i) \in \mathbb{C}[S_n]$, for $i = 1 \ldots n$.
   (b) By looking at eigenvalues of the Jucys-Murphy elements and using the correspondence with content vectors, figure out which partition of n this representation corresponds to.

(2) A poset τ is a *rooted tree* if it has a unique minimal element and its Hasse diagram has no cycles. If $v \in \tau$, define its *hook* to be $H_v := \{w \in \tau \mid w \geq v\}$ with corresponding *hook length* $h_v = |H_v|$. Suppose τ has n nodes. A *natural labeling* (or *linear extension*) of τ is a labeling of the nodes with the numbers $1, \ldots, n$ in a way which is compatible with the partial order; that is, if $u > v$ in the poset, the label of u must be larger than the label of v. Let f^τ denote the number of natural labelings of τ. Give a probabilistic proof of the following "baby hooklength formula":

$$f^\tau = \frac{n!}{\prod_{v \in \tau} h_v}$$

(3) Is the Gelfand-Tsetlin basis of V^λ the same as the basis we constructed earlier using tabloids? If it is, explain why. If not, give an example that illustrates they are different.

(4) A *skew shape* λ/μ is the set-theoretic difference of two Young diagrams λ and μ where $\mu \subset \lambda$. (Align the Young diagrams so that their north-west corners coincide and then remove all boxes of μ.) A standard tableau of a skew shape with n boxes is a filling of the boxes of the shape with $1, \ldots, n$ such that the numbers increase from left to right in each row and from top to bottom in each column. Suppose that λ has n boxes and μ has m boxes. Show that the multiplicity of the representation V^μ in $\mathrm{Res}^{S_n}_{S_m} V^\lambda$ is equal to the number of standard tableau of skew shape λ/μ.

(5) To any permutation $w = v_1 \ldots v_n$ one can assign an *up-down* sequence, which is a sequence of $n-1$ plus or minus signs, the ith being $+$ if $v_i < v_{i+1}$ and $-$ if $v_i > v_{i+1}$. Shown that $Q(w)$ determines the up-down sequence of w. Hint: analyze the insertion

paths that are formed when we insert two successive entries into a tableau.

(6) Prove that the number of paths of length $2n$ in the Young lattice that start and end at the 0th level (and can have up and down edges in any order) equals $(2n-1)!! := 1 \cdot 3 \cdot \cdots \cdot (2n-1)$.

The following fact might be useful: the number of complete matchings of the set $\{1, 2, \ldots, 2n\}$ (i.e. the number of ways to subdivide this set into n pairs) equals $(2n-1)!!$. But if you use this fact then you should prove it.