MATH 155 PROBLEM SET 1 (DUE THURSDAY SEPT. 27)

(1) Prove (using only the definition of representations) that the symmetric group S_n, $n \geq 2$, has exactly two one-dimensional representations: the trivial representation and the sign representation.

(2) Let G be a finite group and let V and W be two representations of G. If ϕ is a G-linear map from V to W, show that $\ker \phi$, $\operatorname{im} \phi$, and $\operatorname{coker} \phi$ are also representations of G.

(3) Recall the definition of the regular representation of G: we take a vector space V with basis $\{e_g \mid g \in G\}$ and let G act on V by $g \cdot \sum a_h e_h = \sum a_h e_{gh}$. Alternatively, let R be the space of complex-valued functions on G, and let $g \in G$ act on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$. Verify that V and R are really the same representation, by identifying the element e_g with the characteristic function which takes the value 1 on g and 0 on other elements of G.

(4) Let G by the cyclic group of order 3, i.e. $G = \{x \mid x^3 = 1\}$, and let V be the two-dimensional G-module with basis v_1, v_2, where

$$x \cdot v_1 = v_2, \quad x \cdot v_2 = -v_1 - v_2.$$

  (a) Show that V is indeed a G-module and write out explicitly the homomorphism $\rho : G \to \mathrm{Gl}_3(\mathbb{C})$.
  (b) Express V as a direct sum of irreducible G-modules.

(5) Let G be a finite group and let $\rho : G \to \mathrm{Gl}_2(\mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

(6) Let G be the infinite group consisting of matrices of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where $n \in \mathbb{Z}$, and let V be the G-module $\mathbb{C}^2$, with the natural multiplication by elements of G (so that for $v \in V$, $g \in G$, the vector gv is just the product of the matrix g with the column vector v). Show that V is not completely reducible. Note that this shows that Maschke's Theorem fails for infinite groups.