155 Lecture 9

Our analysis of the rep theory of S_n so far.

- Prove that $\mathrm{Res}^{S_n}_{S_{n-1}} V$ of an irrep V of S_n has simple multiplicities. ✓

- Using this fact, inductively construct "natural" basis v_T for $\bigoplus_{\lambda \in S_n^\vee} V^\lambda$ ✓

- Define the $GT(n)$ algebra, a maximal comm. subalgebra of $\mathbb{C}[S_n]$ which is diagonal matrices w/ respect to the v_T. $(GT(n) = \langle Z(1), Z(2), \ldots, Z(n) \rangle)$ ✓

- Find nice generators $\{X_i \mid 1 \le i \le n\}$ for $GT(n)$ ✓
  $X_i = (1\ i) + (2\ i) + \cdots + (i-1\ \ i)$

- Compute the eigenvalues of the X_i on each v_T (call this collection $\alpha(T)$). Realize that the set $\mathrm{Spec}(n)$ of all vectors $\alpha(T)$ can be described as the <u>content</u> <u>vectors</u> of std tableaux!

↑ need to do today!

Def: We say that $\alpha = (a_1, \ldots, a_n) \in \mathrm{Cont}(n)$ if α satisfies the following conditions:

(1) $a_1 = 0$

(2) $\{a_q - 1, a_q + 1\} \cap \{a_1, \ldots, a_{q-1}\} \neq \emptyset$ for all $q > 1$

(3) If $a_p = a_q = a$ for some $p < q$ then $\{a-1, a+1\} \subset \{a_{p+1}, \ldots, a_{q-1}\}$

(i.e. between 2 occurrences of a in a content vector there should also be occurrences of $a-1$ and $a+1$)

Last time we observed that $\alpha \in \mathrm{Cont}(n)$ iff α is the content vector of a standard tableau:

Take $T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$ then since $\mathrm{cont}(\lambda) = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 \\ \cline{1-2} -2 \\ \cline{1-1} \end{array}$

we get content vector

$$\alpha(T) = (0, -1, 1, -2, 0, 2, 3).$$

Let $\alpha(v) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ be the eigenvalues of $X_1, \ldots, X_n$ on v where v in GT basis. We call $\alpha(v)$ the <u>weight</u> of v.

Let $\mathrm{Spec}(n) = \{\alpha(v) \mid v \text{ is in the GT basis}\}$
And write v_α for element of GT basis s.t. $\alpha = \alpha(v)$.
Say $\alpha \sim \beta$ if v_α & v_β in same irrep of S_n.

Theorem: $\mathrm{Spec}(n) = \mathrm{Cont}(n)$, where $\mathrm{Spec}(n)$ is the spectrum of $GT(n)$ w/ respect to the YJM-generators $X_1, \dots, X_n$.

To do so, need to understand action of S_n on v_T.

Prop: For any vector v_T, $T = \lambda_0 \nearrow \dots \nearrow \lambda_n$, $\lambda_i \in \hat{S_i}$, and any $k = 1, \dots, n-1$, the vector $s_k \cdot v_T$ is a linear combination of the vectors

$(k\ k+1)$ $\quad v_{T'}, \quad T' = \lambda_0' \nearrow \dots \nearrow \lambda_n' \quad (\lambda_i' \in \hat{S_i})$

s.t. $\lambda_i' = \lambda_i, \quad i \neq k$.

(So the action of s_k affects only the kth level of the branching graph.)

Proof: Let $i > k$. Since $s_k \in S_i$ & the module $\mathbb{C}[S_i] \cdot v_T$ is irred, we have

$$\mathbb{C}[S_i]\, s_k \cdot v_T = \mathbb{C}[S_i] \cdot v_T = V^{\lambda_i},$$

where V^{λ_i} is the irred S_i-module indexed by $\lambda_i \in \hat{S_i}$.

For $i<k$, s_k commutes w/ S_i. So as representations,

$$\mathbb{C}[S_i]s_k . v_T = s_k \mathbb{C}[S_i].v_T \cong \mathbb{C}[S_i].v_T = V^{\lambda_i}$$

So 3 holds for $i<k$ also.

Recall that if $T = \lambda_0 \nearrow \cdots \nearrow \lambda_i \nearrow \cdots \nearrow \lambda_n$, then $\mathbb{C}[S_i].v_T = V^{\lambda_i}$ for $i=1,2,\dots,n$.

Can write $s_k . v_T$ as linear comb of vectors $v_{T'}$. Since we have is

$\mathbb{C}[S_i].(s_k . v_T) = V^{\lambda_i}$ for $i \neq k$, the i^{th} got

part of any such T' must be λ_i. ∎

Can similarly show that the coeff's of this linear comb. depend only on $\lambda_{k-1}, \lambda_k, \lambda_k', \lambda_{k+1}$.

So action of s_k affects only the k^{th} level + depends only on levels $k-1$, k, $k+1$ of branching graph. "local"

Lemma: $s_i X_i + 1 = X_{i+1} s_i$.

Can rewrite this as $s_i X_i s_i + s_i = X_{i+1}$.

Proof: easy check

Action of YJM elmts also local:

Since X_i = sum of transpositions in S_i
– " " " S_{i-1},

a_k: how X_k acts on v_T ...

previous prop $\Rightarrow$

Lemma: If $T = \lambda_0 \nearrow \cdots \nearrow \lambda_n$ and $\alpha(T) = (a_1, \dots, a_{n-1})$ then a_k is the difference of a function of λ_k & a function of λ_{k-1} $\forall k$.

Let $H(2)$ be the algebra generated by the elements Y_1, Y_2, and S subject to relations:

$$s^2 = 1, \quad Y_1 Y_2 = Y_2 Y_1, \quad sY_1 + 1 = Y_2 s.$$

This is a "degenerate affine Hecke algebra"

Remark: Irred. f.dim reps of $H(2)$ have dim 1 or 2

Pf: Since Y_1 & Y_2 commute they have a common eigenbasis in any irrep V. Taking any vector v of V & applying the involution s to v, we get an $H(2)$-invariant subspace of dim 2.

$H(2)$ is important because:

Prop: $\mathbb{C}[S_n]$ is generated by the algebra $\mathbb{C}[S_{n-1}]$ & the algebra $H(2)$ w/ generators $Y_1 = X_{n-1}$, $Y_2 = X_n$, and $s = (n-1\ n)$.

obvious because only need S_n

This prop. allows us to use induction in study of reps of $\mathbb{C}[S_n]$: each step from $n-1$ to n reduces to study of reps of $H(2)$.

Easy to classify reps of $H(2)$:

Because Y_1 & Y_2 have common eigenbasis, each irrep has a vector v s.t. $Y_1 v = av$, $Y_2 v = bv$, $a, b \in \mathbb{C}$

$$\begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} =$$

If v and sv are indep, the relation $sY_1 + 1 = Y_2 s$
$\Rightarrow$ Y_1 and Y_2 act in the basis $\langle v, sv \rangle$ as :

$$Y_1 = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}, \quad Y_2 = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

<u>Exercise</u>: If $b = a \pm 1$, this rep contains the unique one-dim subrep

$$Y_1 \mapsto a, \quad Y_2 \mapsto b, \quad s \mapsto \pm 1$$

Conversely, if v and sv are proportional, then $sv = \pm v$ and $b = a \pm 1$.

<u>Note</u>: we will always have $a \neq b$ since otherwise we couldn't diagonalize the matrices of Y_1, Y_2 (but we know that they are diag in GT basis)

When $a \neq b$, we can diagonalize Y_1 & Y_2:

$$\circledast \quad Y_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad Y_2 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} \frac{1}{b-a} & 1 - \frac{1}{(b-a)^2} \\ 1 & \frac{1}{a-b} \end{pmatrix}$$

(using basis $\{v, \; sv - \frac{1}{b-a} v\}$)

Summarizing, we get:

<u>Prop</u>: Let $\alpha = (a_1, \dots, a_i, a_{i+1}, \dots, a_n) \in Spec(n)$. Then $a_i \in \mathbb{Z}$ and:

(1) $a_i \neq a_{i+1}$ $\forall i$ by <u>Note</u>
(2) If $a_{i+1} = a_i \pm 1$ then $s_i \cdot v_\alpha = \pm v_\alpha$ by <u>Exercise</u>

(3) If $a_{i+1} \neq a_i \pm 1$ then can check that if we define
$$v_{\alpha'} := \left(s_i - \frac{1}{a_{i+1}-a_i}\right) v_\alpha ,$$
the elements s_i, X_i, X_{i+1} act in the basis $\langle v_\alpha, v_{\alpha'} \rangle$ by the matrices ⊛
(replacing Y_1 w/ X_i, Y_2 w/ X_{i+1}, $a+b$ w/ a_i & a_{i+1})

$\therefore$
$\alpha' = s_i \cdot \alpha = (a_1, \dots, a_{i+1}, a_i, \dots, a_n) \in \mathrm{Spec}(n)$
(in $\mathrm{Spec}(n)$ because v_α an eigenvector for the X_i's)

We call any such transposition s_i as above s.t. $s_i \cdot \alpha \in \mathrm{Spec}(n)$ an <u>admissible</u> transposition.
ie. s_i admissible if $a_{i+1} \neq a_i \pm 1$

<u>Next</u>: Show $\mathrm{Spec}(n) \subset \mathrm{Cont}(n)$.

<u>Lemma</u>: Let $\alpha = (a_1, \dots, a_n)$ and $a_i = a_{i+2} = a_{i+1} - 1$ for some i, ie. α contains a fragment of the form $(a, a+1, a)$. Then $\alpha \notin \mathrm{Spec}(n)$

<u>Proof</u>: Let $\alpha \in \mathrm{Spec}(n)$. By (2) of previous prop,
$s_i v_\alpha = v_\alpha$, $\quad s_{i+1} v_\alpha = -v_\alpha$.

So $s_i s_{i+1} s_i v_\alpha = -v_\alpha$ but $s_{i+1} s_i s_{i+1} v_\alpha = v_\alpha$,
contradicting fact that $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

<u>Theorem</u>: $\mathrm{Spec}(n) \subset \mathrm{Cont}(n)$

Recall $\mathrm{Cont}(n) = \{(a_1, \dots, a_n)\}$ s.t. :

(1) $a_1 = 0$

(2) $\{a_q - 1, a_q + 1\} \cap \{a_1, .., a_{q-1}\} \neq \emptyset$ for all $q > 1$

(3) If $a_p = a_q = a$ for some $p < q$ then $\{a-1, a+1\} \subset \{a_{p+1}, \dots, a_{q-1}\}$

Proof: Since $X_1 = 0$, we always have $a_1 = 0$.
We prove (2) and (3) by induction on n.
For $n = 2$ it is trivial, because S_2 has only 2 1-dim rep's w/ eigenvalues ± 1.
So (2) + (3) satisfied.

Assume now $\{a_n - 1, a_n + 1\} \cap \{a_1, \dots, a_{n-1}\} = \emptyset$.
(Recall S_i admissible if $a_{i+1} \neq a_i \pm 1$)
So S_{n-1} is admissible so

$(a_1, \dots, a_{n-2}, a_n, a_{n-1}) \in \mathrm{Spec}(n)$ eigenvalues on $S_{n-1} \cdot v_T$

$\therefore$ $(a_1, \dots, a_{n-2}, a_n) \in \mathrm{Spec}(n-1)$ + clearly
$\{a_n - 1, a_n + 1\} \cap \{a_1, \dots, a_{n-2}\} = \emptyset$.
Contradiction to induction hypothesis.
$\therefore$ (2) holds.

For (3): Assume $a_p = a_n = a$ for some $p < n$ + assume $a - 1 \notin \{a_{p+1}, \dots, a_{n-1}\}$.
Can assume p is as large as possible, i.e.
a doesn't occur between a_p and a_n:
$a \notin \{a_{p+1}, \dots, a_{n-1}\}$.
By induction hypothesis, the # $a+1$ occurs in the set $\{a_{p+1}, \dots, a_{n-1}\}$ at most once:
because if it

occured twice, then by induction, a would also occur.

So either $(a_p, \dots, a_n) = (a, *, \dots, *, a)$ ← no $a+1$

OR

$(a_p, \dots, a_n) = (a, *, \dots, *, a+1, *, \dots, *, a)$.

In Case 1, since there is no $a+1$ or $a-1$ among $*$'s, we can apply $n-p-1$ admissible transpositions to get $\alpha \sim \alpha' = (\dots, a, a, \dots)$, contradicting Claim 1 of previous prop.

In Case 2, applying transpositions yield
$\alpha \sim \alpha' = (\dots, a, a+1, a, \dots)$,
contradicting the previous lemma.

$\therefore$ $\mathrm{Spec}(n) \subset \mathrm{Cont}(n)$. ∎