155 Lecture 7

Point brought up by Charles: std tableaux & construction of v_T are kind of weird. How could we have discovered these constructions?

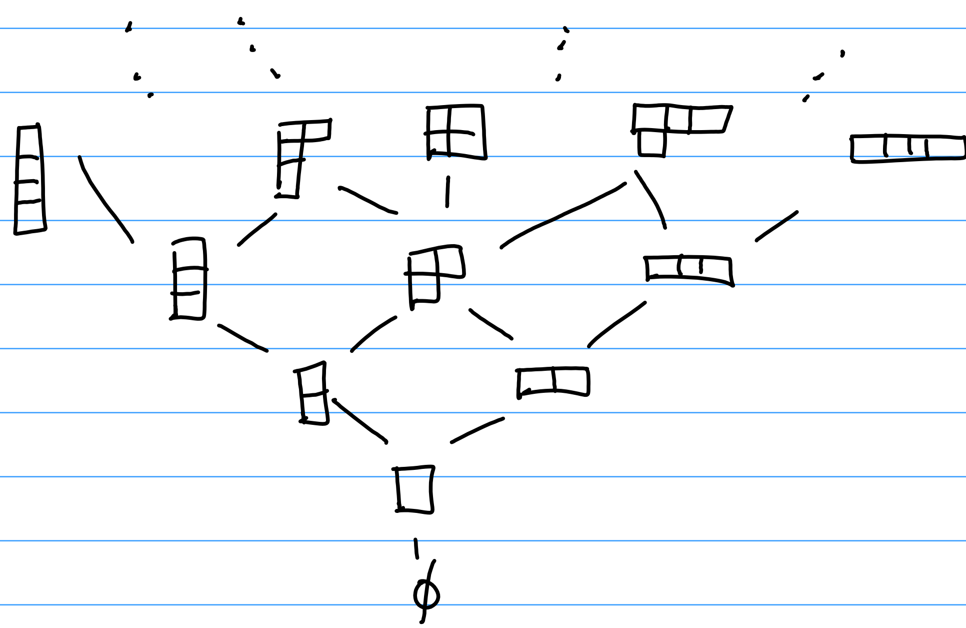
Will give another approach to rep theory of S_n (due to Vershik + Okounkov)
This will allow us to "discover" std tableaux

<u>Def</u>: A poset (partially ordered set) is a set w/ a partial order relation.

<u>Eg</u>: The set of all partitions, ordered by containment of Young diagrams. Called <u>Young's lattice</u>.

We often depict a poset w/ a <u>Hasse diagram</u>.

Eg.



Edges denote cover relations in the poset, i.e. relations $x<y$ s.t. $\nexists$ z s.t. $x<z<y$. We sometimes use $x \lessdot y$ to denote a cover relation.

A chain in a poset is a sequence $x_1 < x_2 < \ldots < x_i$. We call it saturated if $x_1 \lessdot x_2 \lessdot \ldots \lessdot x_i$.

Obs: Saturated chains from $\emptyset$ to λ in Young's lattice are in bijection w/ std tableaux of shape λ!

Ask why!

Idea: Look at chain from top to bottom; 2 adjacent partitions differ by a single box, so as we go down we put $n, n-1, \ldots, 1$ into the boxes that are chopped off.

Question: Consider an irred. rep of S_n. We can restrict this rep to S_{n-1}, i.e. consider it as a rep of S_{n-1}. It may no longer be irred; how does it decompose in terms of irreps of S_{n-1}?

This question leads to an inductive approach to rep theory of S_n, based on fact that there are natural embeddings

$$S_1 \subset S_2 \subset \ldots \subset S_n.$$

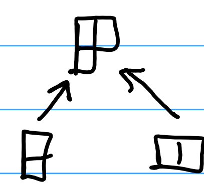
Let $S_n^\vee$ denote the equiv. classes of irreps of S_n. Define the <u>branching graph</u> (or <u>Bratteli diagram</u>) to be a directed graph whose vertices are elemts of $\bigcup_{n \geq 0} S_n^\vee$.

If $\lambda \in S_n^\vee$ (here λ is just a symbol — we are pretending we don't know rep theory of S_n) let V^λ be the S_n-rep corresp. to λ.

Consider decomp. of $V^\lambda|_{S_{n-1}}$.

Let $\mu \in S_{n-1}^\vee$. If V^μ appears k times here, we draw k directed edges from $\mu \in S_{n-1}^\vee$ to $S_n^\vee$ (i.e. $k = \dim \mathrm{Hom}_{S_{n-1}}(V^\mu, V^\lambda)$)

Eg if $V^{\ydiagram{2,1}}|_{S_2} = V^{\ydiagram{1,1}} \oplus V^{\ydiagram{2}}$, draw



We call $S_n^\wedge$ the n^{th} <u>level</u> of the branching graph.

Write $\mu \nearrow \lambda$ if μ & λ connected by edge, & $\mu < \lambda$ if μ & λ connected by chain.

Let ϕ denote unique elemt of $S_0^\wedge$.

We will prove that for S_n, the branching graph is simple — ie. the # of edges from $\mu \in \hat{S_{n-1}}$ to $\lambda \in \hat{S_n}$ is 0 or 1.

This will mean that for $\lambda \in \hat{S_{n-1}}$, the decomposition

$$\circledast \quad V^\lambda = \bigoplus_{\substack{\mu \in \hat{S_{n-1}} \\ \mu \nearrow \lambda}} V^\mu \qquad \text{is } \underline{\text{canonical}}.$$

(no ambiguity about which copy of V^μ we are talking about since only one from each equiv class)

Now take each V^μ on RHS of $\circledast$ & restrict to $\hat{S_{n-2}}$, decomposing into irreps for $\check{S_{n-2}}$. If we keep doing this, we eventually get a canonical decomp. of V^λ into irred. S_1-modules, ie. 1-dim subspaces:

$$V^\lambda = \bigoplus_T V_T \quad \text{indexed by all}$$

possible chains $T = \lambda_0 \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_n$ where $\lambda_i \in \hat{S_i}$ and $\lambda_n = \lambda$.
These are saturated chains in the branching graph.

Choosing a unit vector v_T in each one-dim space V_T, we get a basis $\{v_T\}$ in the module V^λ, called the <u>Gelfand-Tsetlin</u> basis.

* Detour to explain the group algebra *

<u>Recall</u>: $\mathbb{C}[G]$ is the group algebra of G. As a vector space, it has the basis $\{w \mid w \in G\}$. We can add gp elemts formally. ie. $\mathbb{C}[G] = \{ \sum_{w \in G} a_w w : a_w \in \mathbb{C} \}$.

Also we can multiply:

$$\left(\sum_{w \in G} a_w w \right) \left(\sum_{v \in G} b_v v \right) = \sum_{w \in G} \sum_{v \in G} a_w b_v \, wv$$

<u>Def</u>: A <u>rep</u> of $\mathbb{C}[G]$ on v.s. V is a h'ism $\mathbb{C}[G] \to \mathrm{End}(V)$.

So a rep. V of $\mathbb{C}[G]$ is just a $\mathbb{C}[G]$-module. Any rep $\rho : G \to \mathrm{Aut}(V) = GL(V)$ extends by linearity to a map $\tilde{\rho} : \mathbb{C}[G] \to \mathrm{End}(V)$.

ie. define $\tilde{\rho}\left(\sum_{w \in G} a_w w \right) := \sum_{w \in G} a_w \rho(w)$

So rep's of $\mathbb{C}[G]$ correspond exactly to reps of G. If $V = \mathbb{C}[G]$, the $\mathbb{C}[G]$-module structure on $\mathbb{C}[G] \longleftrightarrow$ regular rep.

<u>Recall:</u> The regular rep R decomposes as

$$R = \bigoplus_i (W_i)^{\oplus \dim W_i}$$

i ← varies over all irreps of G

<u>Prop:</u> As <u>algebras</u>, $\mathbb{C}[G] \cong \bigoplus \mathrm{End}(W_i)$

<u>Proof:</u> For any rep W_i of G, the map $G \to \mathrm{Aut}(W_i)$ extends by linearity to a map $\mathbb{C}[G] \to \mathrm{End}(W_i)$. Adding all of these maps together gives us a map $\varphi: \mathbb{C}[G] \longrightarrow \bigoplus \mathrm{End}(W_i)$

This is injective because the regular rep is faithful. Since both sides have dimension $\sum (\dim W_i)^2$, map is $\cong$

* End of detour *

<u>Obs:</u> If $T = \lambda_0 \to \lambda_1 \to \dots \to \lambda_n$ then for all i, $\mathbb{C}[S_i] \cdot v_T$ is the irred. S_i-module V^{λ_i}.

Why? ?

By def, v_T is one of the vectors we get when we restrict V^{λ_i} to S_{i-1} then S_{i-2} ...

Since V^{λ_i} is an irred S_i-module containing v_T, $\mathbb{C}[S_i] \cdot v_T = V^{\lambda_i}$

($v_T \in V^{\lambda_i} \Rightarrow \mathbb{C}[S_i] \cdot v_T \subset V^{\lambda_i}$ and since V^{λ_i} irred, we have $=$)

Let $Z(n)$ denote the center of $\mathbb{C}[S_n]$.

$$Z(n) = \{x \in \mathbb{C}[S_n] : xy = yx \quad \forall\, y \in \mathbb{C}[S_n]\}$$

Let $GT(n) \subset \mathbb{C}[S_n]$ be the algebra generated by the subalgebras $Z(1), Z(2), \ldots, Z(n)$.

Claim: $GT(n)$ is commutative.

Proof: Each $Z(i)$ commutes w/ everything in S_i hence commutes w/ $Z(1), \ldots, Z(i-1)$.
$\therefore$ the $Z(i)$'s all commute w/ each other.
+ so $GT(n)$ is comm.

We saw earlier that

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \in S_n^\vee} \mathrm{End}(V^\lambda) = \begin{pmatrix} \boxed{\Sigma} & & 0 \\ & \boxed{\Sigma} & \\ 0 & & \ddots \boxed{\Sigma} \end{pmatrix}$$

Since $GT(n) \subset \mathbb{C}[S_n]$, we can identify it w/ some matrices here

Prop: (1) $GT(n)$ is the alg. of diagonal matrices w/ respect to the Gelfand-Tsetlin basis in each V^λ

(2) $GT(n)$ is a maximal commutative subalg. in $\mathbb{C}[S_n]$

(3) $v \in V^\lambda$ is in the GT basis iff v is a common eigenvector of elemts of $GT(n)$

(4) Each basis element is uniquely determined by eigenvalues of elts of $GT(n)$.

(1) is nontrivial; the others follow easily.

Proof of (1): Define $P_\lambda := \begin{pmatrix} \boxed{0} & & & & & \\ & \boxed{0} & & & 0 & \\ & & \ddots & & & \\ & & & \overset{V^\lambda}{\boxed{\begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix}}} & & \\ & 0 & & & \ddots & \\ & & & & & \boxed{0} \end{pmatrix}$ (*)

$$P_\lambda : \bigoplus_{\mu \in S_n^\vee} V^\mu \longrightarrow V^\lambda$$

projection

$P_\lambda \in Z(n)$ since it commutes w/ any other block diag. matrix

Given $T = \lambda^0 \to \lambda^1 \to \dots \to \lambda^{n-1} \to \lambda$ construct an operator $P_T := \underset{\underset{Z(0)}{\cap}}{P_{\lambda^0}} \, \underset{\underset{Z(1)}{\cap}}{P_{\lambda^1}} \cdots \underset{\underset{G(n)}{\cap}}{P_\lambda} \in GT(n)$

Writing matrices in the GT basis,

$$P_T = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & \ddots & & & \\ & & & \overset{V_T}{1} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

(multiplying matrices in (*) where nonzero block keeps shrinking)

$\Rightarrow$ all diagonal matrices are in $GT(n)$.
Set of all diag matrices is a maximal comm. subalgebra of the alg. of all matrices.

Since $GT(n)$ is comm., $GT(n) = \{\text{diag matrices}\}$ when written in the GT basis.
$\therefore$ (1) & (2) hold.

Clearly GT (basis) = set of all common eigenvectors of $GT(n)$, so (3).
And each basis vector is determined by eigenvalues of the P_T's, $\therefore$ (4).

Now need to show branching graph of S_n is simple (assumed that earlier) i.e. need
$$\mathrm{Res}^{S_n}_{S_{n-1}} V^\lambda \text{ has simple multiplicities.}$$

Def: If $A \supset B$ are algebras, the centralizer
$$Z(A,B) = \{a \in A \mid ab = ba \quad \forall\, b \in B\}$$
(If $B \subset A$, this is center. If $A \neq B$, can be bigger)

Lemma: $M \subset N$ two algebras. Let V be a f.dim irred. rep of N.
(1) $\mathrm{Res}^N_M V$ has simple multiplicities
(2) $Z(N, M)$ is commutative.

Proof: $(2 \Rightarrow 1)$ Let V^μ & V^λ be f. dim irreps of M and N, resp. Consider the $Z(N,M)$-module $\mathrm{Hom}_M(V^\mu, V^\lambda)$.
(By Schur's Lemma, its dimension is multiplicity V^μ in V^λ.)
Can show it is an irred. $Z(N,M)$-module.
$\therefore$ $Z(N, M)$ comm $\Rightarrow$ $\dim \mathrm{Hom}_N(V^\mu, V^\lambda) = 1$.

$(1) \Rightarrow (2)$ also true but we don't need it.

We want to show $\mathrm{Res}^{S_n}_{S_{n-1}} V^\lambda$ has simple multiplicities so need to show
$$Z_{n-1,1} := Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]) \text{ is commutative.}$$

Need a more detailed description of $Z(n-1,1)$.

For $i = 1, 2, \dots, n$, consider the element
$$X_i = (1\ i) + (2\ i) + \dots + (i-1\ \ i) \in \mathbb{C}[S_n]$$

Called <u>Young-Jucys-Murphy element</u> or YJM-elemts.

Note: $X_1 = 0$

Note: X_i = sum of all transpositions in S_i
 − sum " " in S_{i-1}.

So X_i is the difference of an elem^t of $Z(i)$ & an elem^t of $Z(i-1)$ $\Rightarrow$
$X_i \in GT(n)$ $\forall$ $i \le n$.
In particular, the X_i's commute.

Theorem: $Z(n) \subset \langle Z(n-1), X_n \rangle$
(the algebra generated by the center $Z(n-1)$ of S_{n-1} and X_n)

cycle notation

Proof:
$$X_n = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (i\ j) - \underbrace{\sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} (i\ j)}_{\in Z(n-1)}$$

$$\Rightarrow \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (i\ j) \in \langle Z(n-1), X_n \rangle.$$

$$X_n^2 = \sum_{i,j=1}^{n} (i\ n)(j\ n) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (i\ j\ n) + (n-1)\,\mathrm{id}.$$

$$\therefore \sum_{\substack{i,j=1 \\ i\neq j}}^{n} (i\ j\ n) \in \langle Z(n-1), X_n \rangle$$

Adding the elemt $\sum_{\substack{i\neq j\neq k \\ i,j,k=1}}^{n-1} (i\ j\ k)$ which is in $Z(n-1)$,

we get $\sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} (i, j, k) \in \langle Z(n-1), X_n \rangle$

It's clear that this is in $Z(n)$ also

By induction, we can show that for any i, the sum of all i-cycles

$$\sum_{\substack{a_i\text{'s are} \\ \text{distinct } \& \\ \text{range from 1 to } n}} (a_1\ a_2 \ldots a_i) \quad \in \langle Z(n-1), X_n \rangle.$$

Classical result: the center of $\mathbb{C}[S_n]$ is generated by all such sums of i-cycles.

$$\therefore Z(n) \subset \langle Z(n-1), X_n \rangle.$$