Knuth Equivalence + Green's Theorem

Def: $\pi, \sigma \in S_n$ are said to be P-equivalent, written $\pi \overset{P}{\cong} \sigma$, if $P(\pi) = P(\sigma)$.

On S_3 we have:

$\{123\}$ $\{213, 231\}$ $\{132, 312\}$ $\{321\}$

Def: Suppose $x<y<z$. Then if $\pi, \sigma \in S_n$, say $\pi \overset{1}{\cong} \sigma$ if

1. $\pi = x_1 \ldots yxz \ldots x_n$ and $\sigma = x_1 \ldots yzx \ldots x_n$

and $\pi \overset{2}{\cong} \sigma$ if

2. $\pi = x_1 \ldots xzy \ldots x_n$ and $\sigma = x_1 \ldots zxy \ldots x_n$

Relation $\overset{K}{\cong}$ is trans. closure of $\overset{1}{\cong}$ and $\overset{2}{\cong}$

Theorem (Knuth): If $\pi, \sigma \in S_n$ then $\pi \overset{K}{\cong} \sigma$ iff $\pi \overset{P}{\cong} \sigma$

Proof: ($\Rightarrow$) Last time we saw: enough to show that if $\pi \overset{1}{\cong} \sigma$ then $\pi \overset{P}{\cong} \sigma$

Now assume $\pi \overset{1}{\cong} \sigma$.

$\pi = x_1 \ldots yxz \ldots x_n$ and $\sigma = x_1 \ldots yzx \ldots x_n$

Let P be the tableau obtained by inserting the elements before y.

Need to show: $r_z r_x r_y(P) = r_x r_z r_y(P)$.

Idea: the insertion path of y creates a "barrier" s.t. paths for x and z lie to the left & right of this barrier, no matter in which order they are inserted. So since the paths don't intersect, $r_x r_z$ + $r_z r_x$ have same effect.

Induction on # of rows of P. If P has no rows, then $P(zxy) = \begin{matrix} x & y \\ z & \end{matrix} = P(xyz)$

Now let P have $r > 0$ rows & consider $\overline{P} = r_y(P)$. Let k be column where y enters P, bumping elemt y'. Then:

#1 $\overline{P}_{1,j} \leq y < y'$ for all $j \leq k$

#2 $\overline{P}_{1,\ell} > y'$ for all $\ell > k$.

If we insert x next, since $x < y$, x will enter in a column j w/ $j \leq k$.
And if x' is bumped, then $x' < y'$ by #1.
If we insert z into $r_x r_y(P)$ then since $z > y$, by #1, z enters column ℓ for $\ell > k$. The bumped elemt z' satisfies $z' > y'$ by #2.

Now consider $r_x r_z r_y(P)$. We have #1 and #2.
Again, $z > y \Rightarrow$ by #1, z enters colum ℓ for $\ell > k$, bumping the same z' as above.
And $x < y \Rightarrow x$ enters in a column j w/ $j \leq k$, bumping the same x' as above. Note: $x' < y' < z'$.

So in both cases, z and x enter in columns strictly to right & weakly to left of column k — since columns are disjoint, insertions of x & z don't affect each other. So 1^{st} rows of $r_z r_x r_y(P)$ and $r_x r_z r_y(P)$ are equal.

In both cases we displaced same elmts x', y', z' w/ $x' < y' < z'$, so by induction, the remainders of the two tableaux are equal. This proves $(\Longrightarrow)$.

Before $(\Longleftarrow)$, need some def's ...

Def: If P is a tableau, the row word of P is the perm $\pi_P = R_\ell R_{\ell-1} \dots R_1$ where the R_i are the rows of P.

$$\text{Ex: } P = \begin{matrix} 1\,3\,5\,7 \\ 2\,6 \\ 4 \end{matrix} \Rightarrow \text{row word is } 4\ 2\ 6\ 1\ 3\ 5\ 7$$

Lemma 3: If P a std tableau, then

$$\pi_P \xrightarrow{RS} (P, -) \qquad \text{(Exercise)}$$

$$4\,2\,6\,1\,3\,5\,7 \rightarrow \begin{matrix} 1\ 3\ 5\ 7 \\ 2\ 6 \\ 4 \end{matrix}$$

Note: def's & theorems about std tableaux make sense for partial tableaux & partial perms, i.e. bijections $\pi : K \to L$ between 2 sets of pos. integers. If $K = \{k_1 < \dots < k_m\}$, we can write π as

$$\begin{matrix} k_1 & k_2 & \dots & k_m \\ \downarrow & \downarrow & & \downarrow \\ \ell_1 & \ell_2 & \dots & \ell_m \end{matrix} \quad \text{where } \ell_i = \pi(k_i)$$

Can insert the ℓ_i's into P and place k_i's in Q.

Now back to $(\Longleftarrow)$. By Lemma 3, enough to show that if $P(\pi) = P$ then $\pi \overset{K}{\cong} \pi_P$.

Induction on n. $(\pi \in S_n)$

Let x be last elem't of π, so $\pi = \pi' x$ and π' a sequence of $n-1$ letters.

Let $P' = P(\pi')$.

By induction, $\pi' \overset{K}{\cong} \pi_{P'}$

Need to show $\pi' x \overset{K}{\cong} \pi_P$ so by $\uparrow$, enough to show $\pi_{P'} x \overset{K}{\cong} \pi_P$.

Idea: The Knuth relations used to transform $\pi_{P'} x$ into π_P simulate insertion of x into P'.

Let rows of P' be $R_1, \dots, R_\ell$ where $R_1 = p_1 p_2 \cdots p_k$. If x enters P' in column j, then

$$p_1 < \cdots < p_{j-1} < x < p_j < \cdots < p_k.$$

So

$$\begin{aligned}
\pi_{P'} x &= R_\ell \cdots R_2\, p_1 \cdots p_{j-1} p_j \cdots p_{k-1} p_k\, x && x < p_{k-1} < p_k \\
&\overset{1}{\cong} R_\ell \cdots R_2\, p_1 \cdots p_{j-1} p_j \cdots p_{k-1}\, x\, p_k && x < p_{k-2} < p_{k-1} \\
&\overset{1}{\cong} R_\ell \cdots R_2\, p_1 \cdots p_{j-1} p_j \cdots x\, p_{k-1} p_k \\
&\ \ \vdots \\
&\overset{1}{\cong} R_\ell \cdots R_2\, p_1 \cdots p_{j-1} p_j\, x\, p_{j+1} \cdots p_k && p_{j-1} < x < p_j \\
&\overset{2}{\cong} R_\ell \cdots R_2\, p_1 \cdots p_j\, p_{j-1}\, x\, p_{j+1} \cdots p_k \\
&\ \ \vdots \\
&\overset{2}{\cong} R_\ell \cdots R_2\, p_j\, \underbrace{p_1 \cdots p_{j-1}\, x\, p_{j+1} \cdots p_k}_{\uparrow} && p_{j-2} < p_{j-1} < p_j
\end{aligned}$$

Now this is the first row of $P = r_x(P')$.
And the elemt bumped from first row is just after R_2. So we can continue our sequence of $\overset{1}{\cong}$ and $\overset{2}{\cong}$, until we transform the word into the row word of $r_x(P')$. $=P$

$$\therefore \quad \pi_{P'} X \overset{K}{\cong} \pi_P \qquad \blacksquare$$

Green generalized Schensted's result on increasing + decreasing sequences.

<u>Def:</u> Let π be a sequence. A subsequence σ of π is <u>k-increasing</u> if, as a set, it can be written as the disjoint union

$\sigma = \sigma_1 \cup^* \sigma_2 \cup^* \ldots \cup^* \sigma_k$ where the σ_i are increasing subsequences of π. If the σ_i are all decreasing, call σ <u>k-decreasing</u>.

Let $i_k(\pi)$ = length of π's longest k-increasing subsequences.

$d_k(\pi)$ = " " " k-decreasing subsequences.

$k=1$ gives usual increasing + decreasing sequences.

Ex: $\pi = 4\underline{2}\underline{3}6\underline{5}1\underline{7}$

Longest 1, 2 and 3-increasing subsequences are:

- 2357
- $4236517 = 2357 \cup^* 46$
- $4236517 = 2357 \cup^* 46 \cup^* 1$

So $i_1(\pi) = 4$, $i_2(\pi) = 6$, $i_3(\pi) = 7$

Recall $P(\pi) = \begin{matrix} 1357 \\ 26 \\ 4 \end{matrix}$

Guess the relation of i_j's to shape of $P(\pi)$?

$\lambda = (4, 2, 1)$ so $\lambda_1 = 4$
$\lambda_1 + \lambda_2 = 6$
$\lambda_1 + \lambda_2 + \lambda_3 = 7$

Theorem (Green): Given $\pi \in S_n$, let $\operatorname{sh} P(\pi) = (\lambda_1, \dots, \lambda_\ell)$ w/ conjugate $(\lambda_1', \dots, \lambda_m')$. Then for any k,

$$i_k(\pi) = \lambda_1 + \dots + \lambda_k,$$
$$d_k(\pi) = \lambda_1' + \dots + \lambda_k'.$$

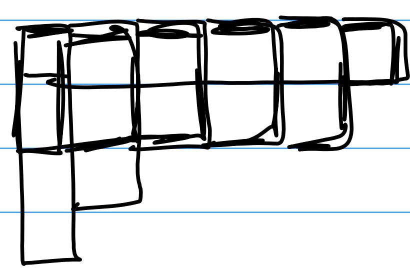
Strategy of proof: given a tableau P, we will prove result for special perm π_P. Then show result holds for all elements in the equiv class of π_P. (all such elements have same shape).

Proof: By def. of π_P, the first k rows of P form a k-increasing subsequence of π_P, so $i_k(\pi_P) \ge \lambda_1 + \lambda_2 + \dots + \lambda_k$.

For reverse inequality, note that: (*) any k-increasing subsequence can intersect a decreasing subsequence in at most k elem'ts. Note: columns of P partition π_P into decreasing subsequences. $\therefore$ if we intersect our k-increas sequence w/ the union of the decreasing sequences formed by the columns, get:

$$\underbrace{i_k(\pi_P)}_{\substack{\text{Length of}\\ \text{longest } k\text{-increasing}\\ \text{sequence}}} \le \sum_{i=1}^{m} \min(\underbrace{\lambda_i'}_{\substack{\text{Length of}\\ \text{a decreasing}\\ \text{subseqn}}}, \underbrace{k}_{\text{By } (*)}) = \sum_{i=1}^{k} \lambda_i$$

To see this, draw a picture, eg.



$k=2$

$\therefore\; i_k(\pi_P) \le \sum_{i=1}^{k} \lambda_i$ so Theorem true for π_P.

Now need to prove Theorem $\forall\, \pi$ s.t. $\pi \overset{k}{\cong} \pi_P$.

By def of $\overset{k}{\cong}$, $\exists$ perm's $\pi_1, \dots, \pi_j$ s.t.

$\pi = \pi_1 \overset{K}{\cong} \pi_2 \overset{K}{\cong} \cdots \overset{K}{\cong} \pi_j = \pi_p$ where each $\overset{K}{\cong}$ is given by a Knuth relation.
All π_i have same tableau, so need to show they have same value for i_k.

Enough to show if $\pi \overset{i}{\cong} \sigma$ for $i=1$ or 2 then $i_k(\pi) = i_k(\sigma)$. Let's do case $\overset{1}{\cong}$ and leave 2 as exercise.

Suppose $\pi = x_1 \ldots y x z \ldots x_n$ and $\sigma = x_1 \ldots y z x \ldots x_n$.
To show $i_k(\pi) \le i_k(\sigma)$, show that any k-incr. subsequence of π has corresp. k-incr. subsequence in σ of same length.

Let $\pi' = \pi_1 \cup^* \pi_2 \cup^* \ldots \cup^* \pi_k < \pi$.
If x & z are not in the same π_i then the π_i are incr. subsequences of σ so done.

Now suppose x & z both in same π_i, WLOG π_1.

- If $y \notin \pi'$, let $\sigma_1 = \pi_1$ w/ x replaced by y.
  $x < y < z$ so σ_1 still increasing &
  $\sigma' = \sigma_1 \cup^* \pi_2 \cup^* \ldots \cup \pi_k^*$ has same length.

- If $y \in \pi'$, say $y \in \pi_2$ (note: can't have $y \in \pi_1$)

Let π_1' = subsequence of π_1 up to & including x
π_1'' = subsequence which is the rest of π_1 } starts w/ z
π_2' = subsequence of π_2 up to + including y
π_2'' = sub " " which is the rest of π_2

So $\pi_i = \pi_i' \pi_i''$.

Define $\sigma_1 = \overset{x}{\overset{\shortparallel}{\pi_1'}} \pi_2''$ and $\sigma_2 = \pi_2' \overset{z}{\overset{\shortparallel}{\pi_1''}}$.

Increasing because $x < y$ $\qquad$ $y < z$

σ_1 & σ_2 are subsequences of σ since x & z no longer in same component. Length correct, so we've shown $i_k(\pi) \le i_k(\sigma)$.

Why is $i_k(\pi) \ge i_k(\sigma)$?

Since x & z out of order in σ, they can never be in same component of a k-incr. subsequence of σ. So each such subseq. of σ is such a subseq. for π. $\therefore$ $i_k(\pi) = i_k(\sigma)$.

The case w/ $\overset{2}{\cong}$ is similar. $\blacksquare$

<u>Rk:</u> Fact that $i_k(\pi) = \lambda_1 + \ldots + \lambda_k$ doesn't imply we can find a k-incr. subsequence $\pi' = \pi_1 \cup^* \ldots \cup^* \pi_k$ s.t. Length of $\pi_i = \lambda_i$.

<u>Ex:</u> $\pi = \underline{2}\,\underline{4}\,7\,9\,\underline{5}\,1\,3\,\underline{6}\,\underline{8}$

$$P(\pi) = \begin{matrix} 1 & 3 & 4 & 6 & 8 \\ 2 & 4 & 9 & & \\ 7 & & & & \end{matrix}$$

so $i_2(\pi) = 5 + 3 = 8$

But only one 2-increasing subsequence of π of length 8...

If we use 24568, no way to complete it.
Can use 2479... and 1368.

Check w/ class: schedule for December.

Mon	Dec 3	(during section)	in-class presentations
Tues	Dec 4		" "
Thurs	Dec 6		" "
Mon	Dec 10	Section: review for exam	
Tues	Dec 11	class: review for exam	
Thurs	Dec 13	no class	
Tues	Dec 18	In-class exam	

Will everyone be there on Dec 18?

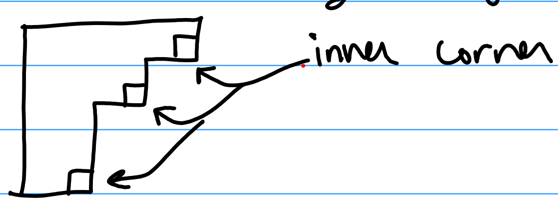
Sketch of proof of hook-length formula

Th: $f^{\lambda} = \dfrac{n!}{\prod_{v \in \lambda} h_v}$

Recall H_v is the hook of v & h_v is its length

Probabilistic proof.
Algorithm due to Greene, Nijenhuis, Wilf.

Let λ be Young diagram w/ n boxes.



GNW 1: Pick a box $v \in \lambda$ w/ prob. $1/n$.
GNW 2: While v is not an inner corner do:
  a. Pick a box $\bar{v} \in H_v - \{v\}$ w/ probability $1/(h_v - 1)$
  b. $v := \bar{v}$

GNW 3: Give the label n to the corner v you have reached.

GNW 4: Go back to step GNW 1 w/ $\lambda := \lambda - \{v\}$ and $n := n-1$ & repeat outer loop until all cells are labeled.

We call the sequence of nodes which we see in one pass thru GNW 1–3 a <u>trial</u>.

Idea: This algorithm produces any given std tableau P of shape λ w/ probability

$$\text{prob}(P) = \frac{\prod_{v \in \lambda} h_v}{n!} \qquad (\star)$$

This would prove the theorem!

<u>Step 1:</u> The algorithm terminates + produces a std tableau.

<u>Step 2:</u> Let (α, β) be the box of P containing n + let $\text{prob}(\alpha, \beta)$ be the prob. that the 1st trial ends there. Prove that $(\star)$ follows by induction from

$$+ \quad \text{prob}(\alpha, \beta) = \frac{1}{n} \prod_{i=1}^{\alpha-1} \left(1 + \frac{1}{h_{i,\beta} - 1}\right) \prod_{j=1}^{\beta-1} \left(1 + \frac{1}{h_{\alpha,j} - 1}\right)$$

Certainly knowing this allows us to compute prob(P)!

<u>Step 3:</u> Given a trial ending at (α, β), let the <u>horiz projection</u> of the trial be

$I = \{ i \neq \alpha : v = (i,j) \text{ for some } v \text{ on the trial} \}$

Similarly define <u>vert projection</u>.

Let $\text{prob}_{I,J}(\alpha,\beta)$ denote the sum of the prob's of all trials terminating at (α,β) w/ horiz + vert projections I & J.

Show: $\text{prob}_{I,J}(\alpha,\beta) = \frac{1}{n} \prod_{i \in I} \frac{1}{h_{i,\beta}-1} \prod_{j \in J} \frac{1}{h_{\alpha,j}-1}$

Use this to show $+$.

Induction on $|I \cup J|$.